

Math 120 Final, Autumn 2007 — Problem 1

(a) Clovis: $(t, C) = (1, 16)$ and $(5, 19)$, slope $\frac{3}{4}$:

$$C(t) = 16 + \frac{3}{4}(t - 1) = 15.25 + 0.75t.$$

(b) Isobel: $(0, 68)$ and $(6, 75)$, slope $\frac{7}{6}$:

$$I(t) = 68 + \frac{7}{6}t.$$

(c) Solve $I = 3C$: $68 + \frac{7}{6}t = 45.75 + 2.25t \Rightarrow 22.25 = (2.25 - \frac{7}{6})t \Rightarrow t \approx 20.54$. So Isobel earns three times as much as Clovis about 20.5 years after 2001, i.e. in 2021 (mid-2021).

Math 120 Final, Autumn 2007 — Problem 2

(a) $y = Ab^t$ with $A = 15$ (the 1982 amount). In 2007 ($t = 25$): $15b^{25} = 8$, so

$$b = \left(\frac{8}{15}\right)^{1/25} \approx 0.97517, \quad y = 15(0.97517)^t.$$

(b) 1996 is $t = 14$: $y = 15(0.97517)^{14} \approx \boxed{10.55 \text{ grams}}$.

(c) $15b^t = 2 \Rightarrow t = \frac{\ln(2/15)}{\ln b} \approx 80.13$, so $\boxed{\text{in 2062}}$ (early 2062).

Math 120 Final, Autumn 2007 — Problem 3

A linear-to-linear function: $S(t) = \frac{at+b}{ct+1}$ with $S(0) = 4$, $S(3) = 10$, $S(6) = 22$.

$S(0) = 4 \Rightarrow b = 4$. $S(3) = 10 \Rightarrow 3a + 4 = 10(3c + 1) \Rightarrow a = 10c + 2$. $S(6) = 22 \Rightarrow 6a + 4 = 22(6c + 1)$, so $60c + 16 = 132c + 22 \Rightarrow c = -\frac{1}{12}$, $a = \frac{7}{6}$:

$$S(t) = \frac{\frac{7}{6}t + 4}{1 - \frac{t}{12}} = \frac{14t + 48}{12 - t}.$$

(Check: $S(3) = 90/9 = 10$, $S(6) = 132/6 = 22$.) Set $S(t) = 202$: $14t + 48 = 2424 - 202t \Rightarrow 216t = 2376 \Rightarrow t = 11$.

Answer: on their 11th anniversary (11 years after marrying).

Math 120 Final, Autumn 2007 — Problem 4

(a) Juliet goes from $(10, 20)$ to the origin in 5 s:

$$x_J = 10 - 2t, \quad y_J = 20 - 4t.$$

(b) Romeo goes from $(30, 0)$ to the origin in 2 s:

$$x_R = 30 - 15t, \quad y_R = 0.$$

(c) Squared distance:

$$D(t) = (x_J - x_R)^2 + (y_J - y_R)^2 = (13t - 20)^2 + (20 - 4t)^2 = 185t^2 - 680t + 800.$$

This quadratic is smallest at $t = \frac{680}{2 \cdot 185} = \frac{68}{37} \approx \boxed{1.84 \text{ s}}$ (they are then about 13.2 m apart).

Math 120 Final, Autumn 2007 — Problem 5

Place the start at the origin with East = $+x$, North = $+y$.

- $0 \leq t \leq 3$: position $(2t, 0)$, distance $2t$.
- $3 \leq t \leq 4$: she is at $(6, 0)$ then walks north: $(6, 3(t - 3))$.
- $4 \leq t \leq 8$: from $(6, 3)$ east at 5 km/h: $(6 + 5(t - 4), 3) = (5t - 14, 3)$.

$$d(t) = \begin{cases} 2t, & 0 \leq t \leq 3 \\ \sqrt{36 + 9(t - 3)^2}, & 3 < t \leq 4 \\ \sqrt{(5t - 14)^2 + 9}, & 4 < t \leq 8 \end{cases} \quad (\text{km, } t \text{ in hours})$$

Math 120 Final, Autumn 2007 — Problem 6

The ground is the line $y = 5x$ (it rises 5 ft per horizontal foot from the bottom of the hill at the origin).

(a) Landing: $-5x^2 + 36x = 5x \Rightarrow x(31 - 5x) = 0 \Rightarrow x = 6.2$. The balloon lands at $\boxed{(6.2, 31)}$.

(b) Height above the ground (vertical distance) is

$$h(x) = (-5x^2 + 36x) - 5x = -5x^2 + 31x,$$

maximized at $x = \frac{31}{10} = 3.1$: $h(3.1) = -5(9.61) + 96.1 = \boxed{48.05 \text{ ft}}$.

Math 120 Final, Autumn 2007 — Problem 7

Max 112 at $t = 2$, next min 88 at $t = 7$: half-period 5, period 10, midline 100, amplitude 12:

$$L(t) = 100 + 12 \cos\left(\frac{2\pi}{10}(t - 2)\right).$$

$L > 94 \iff \cos\left(\frac{\pi}{5}(t - 2)\right) > -\frac{1}{2} \iff \frac{\pi}{5}(t - 2) \in \left(-\frac{2\pi}{3} + 2\pi k, \frac{2\pi}{3} + 2\pi k\right)$, i.e. $t \in \left(2 - \frac{10}{3} + 10k, 2 + \frac{10}{3} + 10k\right)$.
 In $[0, 13]$ that is

$$[0, 5.333) \quad \text{and} \quad (8.667, 13].$$

Total: $5.333 + 4.333 \approx \boxed{9.67 \text{ seconds}}$.

Math 120 Final, Autumn 2007 — Problem 8

(a) In $3 \text{ min} = 180 \text{ s}$ the horse runs 3600 ft, an angle of $\frac{3600}{100} = 36 \text{ rad}$ clockwise. Starting angle is $-\frac{\pi}{2}$ (the point $(0, -100)$), so the final angle is $\theta = -\frac{\pi}{2} - 36$:

$$(100 \cos \theta, 100 \sin \theta) \approx \boxed{(99.18, 12.80)}.$$

(b) The line through $(100, 0)$ with y -intercept 50 is $y = 50 - \frac{x}{2}$. Intersect with $x^2 + y^2 = 10000$:

$$x^2 + \left(50 - \frac{x}{2}\right)^2 = 10000 \Rightarrow \frac{5}{4}x^2 - 50x - 7500 = 0 \Rightarrow x^2 - 40x - 6000 = 0,$$

so $x = 100$ (start) or $x = -60$, giving the far point $(-60, 80)$. Distance: $\sqrt{160^2 + 80^2} = \sqrt{32000} \approx 178.9 \text{ ft}$.
Time at 40 ft/s: $\frac{178.9}{40} \approx \boxed{4.47 \text{ s}}$.

Math 120 Final, Autumn 2007 — Problem 9

From the graph, f is an arch on $[-1, 1]$ with $f(\pm 1) = 0$ and maximum $f(0) = 3$; its range is $[0, 3]$.

(a) $g(x) = -\frac{1}{3}f(2x)$: squeeze horizontally by a factor of 2, then flip over the x -axis and scale vertically by $\frac{1}{3}$. The graph of g is an upside-down arch through $(-\frac{1}{2}, 0)$, down to its minimum $(0, -1)$, and back up to $(\frac{1}{2}, 0)$.

(b) Domain: need $-1 \leq 2x \leq 1$, so $\boxed{-\frac{1}{2} \leq x \leq \frac{1}{2}}$.

(c) Range: $f(2x) \in [0, 3]$, so $g \in \boxed{[-1, 0]}$.

(d) One choice: restrict f to $[0, 1]$, where it decreases from $f(0) = 3$ to $f(1) = 0$ (so it is one-to-one). Its inverse f^{-1} has domain $[0, 3]$ and range $[0, 1]$; its graph is the reflection of that piece across the line $y = x$, running from $(0, 1)$ down to $(3, 0)$. (Restricting to $[-1, 0]$ also works.)