

Math 120 Final, Autumn 2008 — Problem 1

Measure angles counter-clockwise from East. Bert starts at angle 0 with angular speed $\frac{2\pi}{50}$; Ernie starts at $-\frac{\pi}{2}$ going clockwise at $\frac{2\pi}{60}$.

(a) They meet when their angles agree (mod 2π):

$$\frac{2\pi}{50}t = -\frac{\pi}{2} - \frac{2\pi}{60}t + 2\pi k \Rightarrow 2\pi t\left(\frac{1}{50} + \frac{1}{60}\right) = 2\pi k - \frac{\pi}{2}.$$

The first positive time is $k = 1$: $t \cdot \frac{11}{300} = \frac{3}{4}$, so $t = \frac{225}{11} \approx \boxed{20.45 \text{ s}}$.

(b) Bert does a lap (the circumference) in 50 s at 5 ft/s, so the circumference is 250 ft and $r = \frac{250}{2\pi} \approx 39.79$ ft. In 100 s Ernie runs $\frac{100}{60} = 1\frac{2}{3}$ laps, so he is $\frac{2}{3}$ of a lap ($\frac{4\pi}{3}$ rad) from his start. Straight-line distance (chord):

$$2r \sin\left(\frac{2\pi}{3}\right) = r\sqrt{3} \approx \boxed{68.9 \text{ ft}}.$$

Math 120 Final, Autumn 2008 — Problem 2

Write $H(x) = \frac{ax+b}{cx+1}$. $H(0) = 21 \Rightarrow b = 21$. $H(1) = 30 \Rightarrow a + 21 = 30(c + 1) \Rightarrow a = 30c + 9$. $H(10) = 55 \Rightarrow 10a + 21 = 55(10c + 1)$:

$$300c + 111 = 550c + 55 \Rightarrow c = 0.224, \quad a = 15.72.$$

So $H(x) = \frac{15.72x + 21}{0.224x + 1}$. As $x \rightarrow \infty$, $H(x) \rightarrow \frac{a}{c} = \frac{15.72}{0.224} \approx \boxed{70.2 \text{ inches}}$.

Math 120 Final, Autumn 2008 — Problem 3

(a) $V(x) = 2b^x$ with $2b^{10} = 5$, so $V(x) = 2(2.5)^{x/10}$. Quadruple: $(2.5)^{x/10} = 4 \Rightarrow x = \frac{10 \ln 4}{\ln 2.5} \approx \boxed{15.13 \text{ days}}$.

(b) At day 20: $V(20) = 2(2.5)^2 = 12.5$. It drops linearly to 2.5 at day 25: slope $\frac{2.5-12.5}{5} = -2$ dollars/day. Zero when $12.5 - 2(x - 20) = 0 \Rightarrow x = \boxed{26.25 \text{ days}}$.

(c)

$$V(x) = \begin{cases} 2(2.5)^{x/10}, & 0 \leq x \leq 20 \\ 12.5 - 2(x - 20) = 52.5 - 2x, & 20 < x \leq 26.25 \end{cases}$$

Math 120 Final, Autumn 2008 — Problem 4

Put the buoy at the origin. Erik starts at $(-13, 3)$, sails east at 8 mph for 0.75 h (6 miles) to $(-7, 3)$, then sails south along $x = -7$.

East leg ($y = 3$): inside when $x^2 + 9 < 100$, i.e. $|x| < \sqrt{91} \approx 9.539$. He is inside from $x = -9.539$ to $x = -7$: 2.539 miles, taking $\frac{2.539}{8} \approx 0.317$ h.

South leg ($x = -7$): inside when $49 + y^2 < 100$, i.e. $|y| < \sqrt{51} \approx 7.141$. He goes from $y = 3$ down to $y = -7.141$: 10.141 miles, $\frac{10.141}{8} \approx 1.268$ h.

Total time in the radar region: $0.317 + 1.268 \approx \boxed{1.59 \text{ hours}}$.

Math 120 Final, Autumn 2008 — Problem 5

Max 35 at $t = 19.5$, min 25 at $t = 36.5$: half-period 17 h, period 34 h, midline 30, amplitude 5:

$$T(t) = 30 + 5 \cos\left(\frac{2\pi}{34}(t - 19.5)\right).$$

$T > 31 \iff \cos\left(\frac{2\pi}{34}(t - 19.5)\right) > 0.2$. With $\alpha = \cos^{-1}(0.2) \approx 1.3694$, $t \in (19.5 - 7.411 + 34k, 19.5 + 7.411 + 34k)$.
On $[0, 90]$:

$$(12.09, 26.91), (46.09, 60.91), (80.09, 90].$$

Total: $14.82 + 14.82 + 9.91 \approx \boxed{39.6 \text{ hours}}$.

Math 120 Final, Autumn 2008 — Problem 6

Put Juan at the origin. Sven starts at $(-60, 0)$ heading toward $(45, 80)$: direction $\langle 105, 80 \rangle$, length $\sqrt{17425} \approx 132.0$ m. At 3 m/s:

$$x(t) = -60 + \frac{315}{\sqrt{17425}}t \approx -60 + 2.386t, \quad y(t) = \frac{240}{\sqrt{17425}}t \approx 1.818t.$$

(a)

$$d(t) = \sqrt{(-60 + 2.386t)^2 + (1.818t)^2} = \sqrt{9t^2 - 286.36t + 3600}, \quad 0 \leq t \leq 44.0$$

(he reaches $(45, 80)$ at $t = \frac{132.0}{3} \approx 44.0$ s).

(b) $d = 50$: $9t^2 - 286.36t + 1100 = 0 \Rightarrow t = \frac{286.36 \pm \sqrt{42399}}{18}$, so $t \approx 4.47$ s and $t \approx 27.35$ s.

Math 120 Final, Autumn 2008 — Problem 7

Let r be the circles' radius and s the square's side. Wire: $2(2\pi r) + 4s = 100 \Rightarrow s = 25 - \pi r$, with $0 \leq r \leq \frac{25}{\pi}$.
Area:

$$A(r) = 2\pi r^2 + (25 - \pi r)^2 = (2\pi + \pi^2)r^2 - 50\pi r + 625,$$

an upward parabola.

(a) Minimum at the vertex: $r = \frac{50\pi}{2(2\pi + \pi^2)} = \frac{25}{2 + \pi} \approx \boxed{4.86 \text{ in}}$ (area $\approx 243.1 \text{ in}^2$).

(b) The maximum is at an endpoint: $A(0) = 625$, $A(25/\pi) = \frac{1250}{\pi} \approx 397.9$. So the area is largest with $\boxed{r = 0}$ (no circles: all 100 in make one square, 625 in^2).