

Math 120 Final, Winter 2008 — Problem 1

Let t be hours after midnight. Minimum 2000 ft at $t = 3$, maximum 2500 ft at $t = 7.5$: half-period 4.5 h, period 9 h, midline 2250, amplitude 250. Starting from the minimum:

$$A(t) = 2250 - 250 \cos\left(\frac{2\pi}{9}(t - 3)\right).$$

$A < 2100 \iff \cos\left(\frac{2\pi}{9}(t-3)\right) > 0.6$. With $\alpha = \cos^{-1}(0.6) \approx 0.9273$, this means $\frac{2\pi}{9}(t-3) \in (-\alpha+2\pi k, \alpha+2\pi k)$, i.e. $t \in (3 - 1.328 + 9k, 3 + 1.328 + 9k)$. On $[0, 24]$:

$$(1.672, 4.328), \quad (10.672, 13.328), \quad (19.672, 22.328),$$

each 2.656 h long. Total: ≈ 7.97 hours.

Math 120 Final, Winter 2008 — Problem 2

Let t be years after 1980. Gumbolantenden: $G(t) = 20000(1.0107)^t$.

Attalioto: $A(20) = 40000$, $A(25) = 62000$, so its yearly factor is $b = (62000/40000)^{1/5} = 1.55^{1/5} \approx 1.09161$ and

$$A(t) = 40000 b^{t-20}.$$

Set $G(t) = A(t)$ and take logs:

$$\ln 20000 + t \ln 1.0107 = \ln 40000 + (t - 20) \ln b \Rightarrow t = \frac{20 \ln b - \ln 2}{\ln b - \ln 1.0107} \approx \boxed{13.76 \text{ years after 1980}}$$

(both about 23,160 people, in late 1993).

Math 120 Final, Winter 2008 — Problem 3

(a) Nell covers $\langle 1 - 20, 4 - 8 \rangle = \langle -19, -4 \rangle$ in 3 s:

$$x(t) = 20 - \frac{19}{3}t, \quad y(t) = 8 - \frac{4}{3}t.$$

(b) Kit moves along $y = -0.5x + 2.5$ toward the y -axis (decreasing x). Direction $\langle -1, 0.5 \rangle$, length $\sqrt{1.25} = \frac{\sqrt{5}}{2}$, so the unit direction is $\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle$. At 5 ft/s from $(1, 2)$:

$$x(t) = 1 - 2\sqrt{5}t \approx 1 - 4.472t, \quad y(t) = 2 + \sqrt{5}t \approx 2 + 2.236t.$$

Math 120 Final, Winter 2008 — Problem 4

Put Albany at $(0, 0)$ (East = $+x$, North = $+y$). Birmingham is $(7, 10)$ and Cambridge is $(7+3, 10-4) = (10, 6)$.

Bob's position: $(7s, 10s)$, $0 \leq s \leq 1$. Squared distance to Cambridge:

$$D(s) = (7s - 10)^2 + (10s - 6)^2 = 149s^2 - 260s + 136,$$

minimized at $s = \frac{260}{298} = \frac{130}{149} \approx 0.8725$ (inside $[0, 1]$), the point $(6.107, 8.725)$. Closest distance:

$$\sqrt{(10 - 6.107)^2 + (6 - 8.725)^2} \approx \boxed{4.75 \text{ km}}.$$

Math 120 Final, Winter 2008 — Problem 5

Let d be the first distance and h Godzilla's height then. First: $h = d \tan 73^\circ$. Second (120 ft farther, height $h + 24$): $h + 24 = (d + 120) \tan 66^\circ$.

Substitute: $d \tan 73^\circ + 24 = (d + 120) \tan 66^\circ$, so

$$d = \frac{120 \tan 66^\circ - 24}{\tan 73^\circ - \tan 66^\circ} \approx 239.58 \text{ ft}, \quad h \approx 783.6 \text{ ft}.$$

At the second measurement Godzilla was $h + 24 \approx \boxed{807.6 \text{ ft}}$ tall.

Math 120 Final, Winter 2008 — Problem 6

Let s be the side of each square and w the short side of the rectangle (long side $2w$). Fencing (each enclosure fenced separately):

$$3(4s) + 2(w + 2w) = 12s + 6w = 300 \Rightarrow w = 50 - 2s, \quad 0 \leq s \leq 25.$$

Total area:

$$A(s) = 3s^2 + 2w^2 = 3s^2 + 2(50 - 2s)^2 = 11s^2 - 400s + 5000.$$

This parabola opens *upward*, so its vertex $s = \frac{200}{11} \approx 18.2$ ft gives the *minimum* total area (≈ 1364 ft²). On $0 \leq s \leq 25$ the maximum is at an endpoint: $A(0) = 5000$, $A(25) = 1875$.

Answer: the total area is largest when the squares are as small as possible ($s \rightarrow 0$, area approaching 5000 ft², all fencing in the rectangle). If the question intends the vertex of the parabola, note that $s = \frac{200}{11} \approx 18.18$ ft is where the area is smallest, not largest.

Math 120 Final, Winter 2008 — Problem 7

(a) For $x \geq 1$: $|x - 1| = x - 1$, so $f(x) = 2x + 5 - 4x + 4 = -2x + 9$. For $x < 1$: $|x - 1| = 1 - x$, so $f(x) = 2x + 5 - 4 + 4x = 6x + 1$.

$$f(x) = \begin{cases} 6x + 1, & x < 1 \\ -2x + 9, & x \geq 1 \end{cases}$$

(b) If $a \geq 1$: $-2a + 9 = 7a - 8 \Rightarrow a = \frac{17}{9}$, which is ≥ 1 ✓. If $a < 1$: $6a + 1 = 7a - 8 \Rightarrow a = 9$, not < 1 , rejected.

$$\boxed{a = \frac{17}{9}}$$