

Math 124I - Winter 2003
Mid-Term Exam Number One
January 30, 2003
Solutions and Commentary

There were two versions of the exam used. If this doesn't look like yours, try the class website for the other one.

1. Decide whether the following statements are true or false and circle the corresponding letter below the statement. You need not show any work (i.e., these are “all or nothing” problems).

If $\cos \theta = \frac{2}{3}$, then $\sin \theta$ must equal $\frac{\sqrt{5}}{3}$.

T F

Solution: This is false. Since

$$\cos^2 \theta + \sin^2 \theta = 1$$

we may conclude, if $\cos \theta = \frac{2}{3}$, that $\sin \theta = \pm \frac{\sqrt{5}}{3}$.

Polynomials are continuous everywhere that they are defined.

T F

Solution: This is true. This is also true of rational functions, trigonometric functions (e.g., $\sin x$, $\cos x$, $\tan x$), and exponential functions (e.g., e^x).

If $\lim_{x \rightarrow 5^+} f(x) = \infty$ and $\lim_{x \rightarrow 5^-} f(x) = \infty$, then $\lim_{x \rightarrow 5} f(x) = \infty$.

T F

Solution: This is true. We say $\lim_{x \rightarrow 5} f(x) = \infty$ **precisely** when $\lim_{x \rightarrow 5^+} f(x) = \infty$ and $\lim_{x \rightarrow 5^-} f(x) = \infty$.

$$\lim_{x \rightarrow 3} \frac{1}{(x-3)^8} = \infty.$$

T F

Solution: This is true. As x approaches 3 (from either side), $(x-3)^8$ approaches zero, and since 8 is an even power, the value of $(x-3)^8$ stays positive. So $\frac{1}{(x-3)^8}$ grows positively without bound as x approaches 3 (i.e., $\lim_{x \rightarrow 3^+} \frac{1}{(x-3)^8} = \infty$ and $\lim_{x \rightarrow 3^-} \frac{1}{(x-3)^8} = \infty$). Hence, $\lim_{x \rightarrow 3} \frac{1}{(x-3)^8} = \infty$.

$$\lim_{x \rightarrow 3} \frac{1}{(x-3)^7} = \infty.$$

T F

Solution: This is false, since $\lim_{x \rightarrow 3^+} \frac{1}{(x-3)^7} = \infty$ and $\lim_{x \rightarrow 3^-} \frac{1}{(x-3)^7} = -\infty$.

2. Suppose $f(x)$ is defined for all $x \neq 0$, and that $f'(x) = \frac{3}{2}x^2 - \frac{1}{x}$. Find the x -value of each point on the curve $y = f(x)$ at which the tangent line is horizontal.

Solution: Since a horizontal line has a slope of zero, we are interested in when $f'(x)$ equals zero. Setting $f'(x)$ equal to zero and solving for x , we have:

$$f'(x) = \frac{3}{2}x^2 - \frac{1}{x} = 0$$

$$\frac{3}{2}x^3 - 1 = 0$$

$$\frac{3}{2}x^3 = 1$$

$$x^3 = \frac{2}{3}$$

$$x = \sqrt[3]{\frac{2}{3}}$$

So the point on the curve with $x = \sqrt[3]{\frac{2}{3}}$ has a horizontal tangent line.

Commentary: This problem required you to know two things:

- (a) Horizontal lines have zero slopes.
- (b) $f'(x)$ is the slope of the tangent line to $y = f(x)$ at the point $(x, f(x))$.

3. Let

$$f(x) = \begin{cases} x^3 - 5 & \text{if } x < 4 \\ 3x + 47 & \text{if } 4 \leq x \leq 7 \\ x^2 + 29 & \text{if } x > 7 \end{cases}$$

Find all points at which f is discontinuous. Justify your answer.

Solution: Since f is a polynomial in any interval that doesn't include 4 or 7, the function can only be discontinuous at 4 or 7. We check the limits:

$$\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^-} (x^3 - 5) = 4^3 - 5 = 59$$

$$\lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^+} (3x + 47) = 12 + 47 = 59$$

$$\lim_{x \rightarrow 7^-} f(x) = \lim_{x \rightarrow 7^-} (3x + 47) = 21 + 47 = 69$$

$$\lim_{x \rightarrow 7^+} f(x) = \lim_{x \rightarrow 7^+} (x^2 + 29) = 49 + 29 = 78.$$

Since $\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^+} f(x) = f(4)$, f is continuous at $x = 4$.

Since $\lim_{x \rightarrow 7^-} f(x) \neq \lim_{x \rightarrow 7^+} f(x)$, f is **not** continuous at $x = 7$.

Commentary: This problem is almost identical to problem #36 of section 2.5 of the assigned homework.

4. Suppose

$$\frac{f(a+h) - f(a)}{h} = \frac{-3}{(a-3)(a+h-3)}.$$

(a) Find the slope of the secant line through $(2, f(2))$ and $(8, f(8))$.

Solution: The expression

$$\frac{f(a+h) - f(a)}{h}$$

is the slope of the secant line through the points $(a, f(a))$ and $(a+h, f(a+h))$. So, letting $a = 2$ and $h = 6$, we have the slope of the secant line through $(2, f(2))$ and $(8, f(8))$ equal to

$$\frac{f(a+h) - f(a)}{h} = \frac{-3}{(2-3)(2+6-3)} = \frac{-3}{(-1)(5)} = \frac{3}{5}.$$

(b) Find the slope of the tangent line to the curve $y = f(x)$ at the point $(4, f(4))$.

Solution: The slope of the tangent line to curve $y = f(x)$ at the point $(a, f(a))$ is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{-3}{(a-3)(a+h-3)} = \frac{-3}{(a-3)^2}.$$

Letting $a = 4$, we have $f'(4) = -3$.

Commentary: This problem is essentially identical (except for numerical changes) to a problem worked in lecture on 1/27/03. In class on 1/27/03 and again on 1/29/03, I said it would be on the exam.

5. Evaluate the limit:

$$\lim_{x \rightarrow \infty} \frac{\sqrt{(1+2x)(2+3x)}}{3+7x}$$

Solution:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\sqrt{(1+2x)(2+3x)}}{3+7x} &= \lim_{x \rightarrow \infty} \sqrt{\frac{(1+2x)(2+3x)}{(3+7x)^2}} \\ &= \lim_{x \rightarrow \infty} \sqrt{\frac{1+7x+6x^2}{9+42x+49x^2}} = \lim_{x \rightarrow \infty} \sqrt{\frac{\frac{1}{x^2} + \frac{7}{x} + 6}{\frac{9}{x^2} + \frac{42}{x} + 49}} \end{aligned}$$

$$= \sqrt{\frac{6}{49}} = \frac{\sqrt{6}}{7}.$$

Commentary: This problem is very similar to assigned homework problem #17 of section 2.6. The difference is the factored numerator, which you experienced in homework problem #13 of section 2.6.

6. The height (in meters) of an arrow shot straight up from the ground after t seconds is given by

$$h(t) = 60t - 4.9t^2.$$

- (a) Find the average velocity of the arrow over the interval $[1, 3]$.

Solution: Since $h(1) = 60 - 4.9 = 55.1$ and $h(3) = 60(3) - 4.9(3^2) = 135.9$, and $h(3) - h(1) = 80.8$, the arrow moved 80.8 meters in an interval of 2 seconds. So, its average velocity during this time was $\frac{80.8}{2} = 40.4$ meters per second.

Commentary: This problem is virtually identical to problem #6 of section 2.1 of the assigned homework.

- (b) Find a 1-second interval of time over which the arrow's average velocity is 48.24 meters/second.

Solution: For a 1-second interval of time starting at time t , the average velocity of the arrow is $\frac{h(t+1) - h(t)}{t+1-t} = h(t+1) - h(t)$. We need to solve $h(t+1) - h(t) = 48.24$:

$$h(t+1) - h(t) = 60(t+1) - 4.9(t+1)^2 - (60t - 4.9t^2) = 48.24$$

$$60t + 1 - 4.9t^2 + 9.8t + 4.9 - 60t + 4.9t^2 = 48.24$$

$$55.1 - 9.8t = 48.24$$

$$t = 0.7.$$

Commentary: This problem was the one I thought would be the trickiest. It required you to write a general expression for the average velocity of the arrow over a one-second time interval, set that equal to the given value and solve. A variety of methods were used to good effect on this one.

7. Let C be the circle with radius 1 centered at the origin. Find all points on C at which the tangent line to C passes through the point $(10, 0)$.

Solution: The tangent line to a circle is perpendicular to the radius (i.e., the line between the center of the circle and the point of tangency). Suppose (a, b) is the point of tangency.

Since (a, b) is on the unit circle, we know $a^2 + b^2 = 1$.

The slope of the line through $(0, 0)$ and (a, b) is b/a , so the slope of the tangent line is $-a/b$.

Since we want the tangent line to pass through $(10, 0)$, the slope must also be

$$\frac{b - 0}{a - 10} = \frac{b}{a - 10}.$$

Equating these two slope expressions we have

$$-\frac{a}{b} = \frac{b}{a - 10}$$

from which we can derive

$$-a(a - 10) = b^2.$$

Since $a^2 + b^2 = 1$, we have

$$a^2 - a(a - 10) = 1$$

which gives us $a = 1/10$. Solving for b we find $b = \pm \frac{\sqrt{99}}{10}$. Hence the points on C at which the tangent line to C passes through the point $(10, 0)$ are $(\frac{1}{10}, \frac{\sqrt{99}}{10})$ and $(\frac{1}{10}, -\frac{\sqrt{99}}{10})$

Commentary: This problem is similar to problem #1 of Quiz 1. It also uses ideas from Worksheet 1 from the first week of class.