

Math 124 Midterm 2, Autumn 2011 (Loveless) — Problem 1

(a) $f(x) = \ln(\tan^{-1}(x^3))$:

$$f'(x) = \frac{1}{\tan^{-1}(x^3)} \cdot \frac{3x^2}{1+x^6}, \quad f'(1) = \frac{3/2}{\pi/4} = \boxed{\frac{6}{\pi}}.$$

(b) $y = (e^{2x} - \ln x)^{10}$:

$$y' = 10(e^{2x} - \ln x)^9 \left(2e^{2x} - \frac{1}{x} \right), \quad y'(1) = 10(e^2)^9(2e^2 - 1) = \boxed{10e^{18}(2e^2 - 1)}.$$

(c) $y = \frac{1}{2}x^{\sqrt{x}}$. Take logs: $\ln y = \ln \frac{1}{2} + \sqrt{x} \ln x$, so

$$\frac{y'}{y} = \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x} = \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}.$$

At $x = 4$: $y = \frac{1}{2} \cdot 4^2 = 8$ and $\frac{y'}{y} = \frac{\ln 4}{4} + \frac{1}{2}$, so $y'(4) = 8 \left(\frac{\ln 4}{4} + \frac{1}{2} \right) = 2 \ln 4 + 4 = \boxed{4 + 4 \ln 2}$.

Math 124 Midterm 2, Autumn 2011 (Loveless) — Problem 2

y -intercepts: $x = 0$ gives $y^2 = \cos 0 = 1$, so $y = \pm 1$.

Differentiate $x^2 + y^2 = \cos x + 3xy^3$:

$$2x + 2yy' = -\sin x + 3y^3 + 9xy^2y'.$$

At $x = 0$: $2yy' = 3y^3$, so $y' = \frac{3y^2}{2} = \frac{3}{2}$ at both points.

At $(0, 1)$: $y = \frac{3}{2}x + 1$. At $(0, -1)$: $y = \frac{3}{2}x - 1$.

Math 124 Midterm 2, Autumn 2011 (Loveless) — Problem 3

$$f'(x) = 2x - \frac{6x}{x^2 + 1} = \frac{2x(x^2 + 1) - 6x}{x^2 + 1} = \frac{2x(x^2 - 2)}{x^2 + 1}.$$

Critical numbers in $[-1, 3]$: $x = 0$ and $x = \sqrt{2}$ ($-\sqrt{2}$ is outside). Compare:

$$f(-1) = 1 - 3 \ln 2 \approx -1.079, \quad f(0) = 0, \quad f(\sqrt{2}) = 2 - 3 \ln 3 \approx -1.296, \quad f(3) = 9 - 3 \ln 10 \approx 2.092.$$

Absolute maximum $\boxed{9 - 3 \ln 10}$ at $x = 3$; absolute minimum $\boxed{2 - 3 \ln 3}$ at $x = \sqrt{2}$.

Math 124 Midterm 2, Autumn 2011 (Loveless) — Problem 4

(a) $f(x) = \cos x - x$ is continuous with $f(0) = 1 > 0$ and $f(\pi) = -1 - \pi < 0$, so by the Intermediate Value Theorem it has at least one zero in $(0, \pi)$. Also $f'(x) = -\sin x - 1$, and on $[0, \pi]$ we have $\sin x \geq 0$, so $f'(x) \leq -1 < 0$: f is strictly decreasing there and can cross zero only once. So there is exactly one solution.

(b) At $a = \frac{\pi}{4}$: $f(a) = \frac{\sqrt{2}}{2} - \frac{\pi}{4} \approx -0.078291$ and $f'(a) = -\frac{\sqrt{2}}{2} - 1 \approx -1.707107$. Linearization $L(x) = f(a) + f'(a)(x - a)$; setting $L(x) = 0$:

$$x = a - \frac{f(a)}{f'(a)} = 0.785398 - \frac{-0.078291}{-1.707107} \approx \boxed{0.73954}.$$

Math 124 Midterm 2, Autumn 2011 (Loveless) — Problem 5

(a) After 4 min, $V = 4(12\pi) = 48\pi$ ft³, $r = 6$, $h = 4$ (check: $\frac{\pi}{3}(36)(4) = 48\pi$). Differentiate $V = \frac{\pi}{3}r^2h$:

$$\frac{dV}{dt} = \frac{\pi}{3} \left(2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right) \Rightarrow 12\pi = \frac{\pi}{3} \left(2(6)(4) \frac{1}{4} + 36 \frac{dh}{dt} \right).$$

So $36 = 12 + 36h'$, and $\frac{dh}{dt} = \boxed{\frac{2}{3} \text{ ft/min}}$.

(b) Each equal side is $s = \sqrt{r^2 + h^2}$, so $s s' = r r' + h h'$. With $s = \sqrt{52} = 2\sqrt{13}$:

$$s' = \frac{6(\frac{1}{4}) + 4(\frac{2}{3})}{2\sqrt{13}} = \frac{25/6}{2\sqrt{13}} \approx \boxed{0.578 \text{ ft/min}}.$$

Math 124 Midterm 2, Autumn 2011 (Loveless) — Problem 6

(a) Radius $\boxed{70 \text{ ft}}$; at $t = 0$ the location is $\boxed{(70, 74)}$; one revolution when $\frac{\pi}{15}t = 2\pi$, i.e. $\boxed{30 \text{ s}}$.

(b)

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\frac{70\pi}{15} \cos(\frac{\pi t}{15})}{-\frac{70\pi}{15} \sin(\frac{\pi t}{15})} = -\cot\left(\frac{\pi t}{15}\right).$$

At $t = 10$: $-\cot \frac{2\pi}{3} = \boxed{\frac{1}{\sqrt{3}}}$.

(c) The far left point is $(-70, 74)$, reached when $\frac{\pi t}{15} = \pi$ ($t = 15$). There your vertical velocity is $y'(15) = \frac{70\pi}{15} \cos \pi = -\frac{14\pi}{3} \approx -14.66 \text{ ft/s}$ (moving down). The balloon hits a 6-ft head at $t = 1$:

$$h(1) = 74 - A - 16 = 6 \Rightarrow A = 52 \text{ ft/s}.$$

A is the wheel's downward speed plus your throw, so your arm threw it at

$$52 - \frac{14\pi}{3} \approx \boxed{37.3 \text{ ft/s}}.$$