

Math 124 Midterm 2, Autumn 2012 (Loveless) — Problem 1

(a) At $x = 0$: $y = \sqrt{e^0 + \ln 1 + 1} = \sqrt{2}$. Chain rule:

$$y' = \frac{e^{\sin x} \cos x + \frac{5}{5x+1}}{2\sqrt{e^{\sin x} + \ln(5x+1) + 1}}, \quad y'(0) = \frac{1+5}{2\sqrt{2}} = \frac{3}{\sqrt{2}}.$$

Tangent line: $y = \sqrt{2} + \frac{3}{\sqrt{2}}x$.

(b) Differentiate $y^5 - x = yx^2 + 1$:

$$5y^4y' - 1 = y'x^2 + 2xy.$$

At $(0, 1)$: $5y' - 1 = 0$, so $y' = \frac{1}{5}$. Tangent line $y \approx 1 + \frac{1}{5}x$, so at $x = 0.3$:

$$y \approx 1 + \frac{1}{5}(0.3) = \boxed{1.06} \quad (\text{actual} \approx 1.069).$$

Math 124 Midterm 2, Autumn 2012 (Loveless) — Problem 2

(a) f is differentiable everywhere, so by the Mean Value Theorem on $[0, x]$ there is a c in $(0, x)$ with

$$\frac{f(x) - f(0)}{x - 0} = f'(c), \quad \text{i.e.} \quad f(x) = 1 + f'(c)x.$$

Since $2 \leq f'(c) \leq 5$ and $x > 0$:

$$\boxed{1 + 2x \leq f(x) \leq 1 + 5x}.$$

(b)

$$f'(x) = \frac{2x}{1+x^4} - \frac{1}{8} \cdot \frac{4x^3}{x^4+1} = \frac{2x - \frac{1}{2}x^3}{1+x^4} = \frac{x(4-x^2)}{2(1+x^4)}.$$

Critical numbers: $x = 0, \pm 2$. Sign of f' : positive on $(-\infty, -2)$, negative on $(-2, 0)$, positive on $(0, 2)$, negative on $(2, \infty)$.

So $x = -2$ and $x = 2$ are **local maxima**, and $x = 0$ is a **local minimum**.

Math 124 Midterm 2, Autumn 2012 (Loveless) — Problem 3

(a) Quotient rule:

$$f'(x) = \frac{e^x(ax^2 + b) - e^x(2ax)}{(ax^2 + b)^2} = \frac{e^x(ax^2 - 2ax + b)}{(ax^2 + b)^2}.$$

The denominator is never 0 ($a, b > 0$), so the critical numbers solve $ax^2 - 2ax + b = 0$:

$$x = \frac{2a \pm \sqrt{4a^2 - 4ab}}{2a} = 1 \pm \sqrt{1 - \frac{b}{a}}.$$

Two real critical numbers exactly when $4a^2 - 4ab > 0$, i.e. $\boxed{a > b}$.

(b) $\ln f = -\frac{\ln x}{x^2}$. Differentiating:

$$\frac{f'}{f} = -\frac{\frac{1}{x} \cdot x^2 - \ln x \cdot 2x}{x^4} = -\frac{1 - 2\ln x}{x^3}.$$

$f' = 0$ when $\ln x = \frac{1}{2}$, i.e. $x = \sqrt{e} \approx 1.649$ (in the interval). Compare:

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{-4} = 16, \quad f(\sqrt{e}) = e^{-1/(2e)} \approx 0.832, \quad f(2) = 2^{-1/4} \approx 0.841.$$

Absolute maximum $\boxed{16}$ at $x = \frac{1}{2}$; absolute minimum $\boxed{e^{-1/(2e)} \approx 0.832}$ at $x = \sqrt{e}$.

Math 124 Midterm 2, Autumn 2012 (Loveless) — Problem 4

(a) By similar triangles $\frac{r}{h} = \frac{2}{8}$, so $r = \frac{h}{4}$ and

$$V = \frac{\pi}{3} \left(\frac{h}{4} \right)^2 h = \frac{\pi h^3}{48}, \quad \frac{dV}{dt} = \frac{\pi h^2}{16} \frac{dh}{dt}.$$

At $h = 2$, $h' = \frac{1}{3}$: $\frac{dV}{dt} = \frac{4\pi}{16} \cdot \frac{1}{3} = \frac{\pi}{12} \approx 0.2618 \text{ ft}^3/\text{min}$. Also $\frac{dV}{dt} = 0.4 - c$, so

$$c = 0.4 - \frac{\pi}{12} \approx \boxed{0.138 \text{ ft}^3/\text{min}}.$$

(b) The bucket holds $\pi \left(\frac{1}{2} \right)^2 (2) = \frac{\pi}{2} \approx 1.571 \text{ ft}^3$. Dripping at c from $t = 0$, it fills at

$$t = \frac{\pi/2}{0.4 - \pi/12} \approx \boxed{11.4 \text{ min}}.$$

Math 124 Midterm 2, Autumn 2012 (Loveless) — Problem 5

- (a) Northernmost at $t = 0$: $\theta_0 = \frac{\pi}{2}$. Counterclockwise at 5 rev/h: $w = 10\pi$ rad/h. The center moves from $(0, 0)$ to $(4000, 6000)$ in 3 h: $a = \frac{4000}{3}$, $b = 2000$.
- (b) At $t = 4.5$: angle $\frac{\pi}{2} + 45\pi \equiv \frac{3\pi}{2} \pmod{2\pi}$, so

$$x = 6000 + 4 \cos \frac{3\pi}{2} = 6000, \qquad y = 9000 + 4 \sin \frac{3\pi}{2} = 8996.$$

Velocity: $x' = a - 4w \sin(\theta_0 + wt) = \frac{4000}{3} + 40\pi \approx 1458.997$, and $y' = b + 4w \cos(\theta_0 + wt) = 2000$. Slope $\frac{2000}{1458.997} \approx 1.3708$:

$$y - 8996 = 1.3708 (x - 6000).$$