

Math 125 Final, Autumn 2023 — Problem 1

(a) $x^2 + 2x + 10 = (x + 1)^2 + 9$. Write $x = (x + 1) - 1$:

$$\int \frac{x + 1}{\sqrt{(x + 1)^2 + 9}} dx - \int \frac{dx}{\sqrt{(x + 1)^2 + 9}} = \boxed{\sqrt{x^2 + 2x + 10} - \ln \left| x + 1 + \sqrt{x^2 + 2x + 10} \right| + C}$$

(the second integral by $x + 1 = 3 \tan \theta$, giving $\ln |\sec \theta + \tan \theta|$).

(b) $\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$:

$$\int \frac{1}{2}(\sin 6x + \sin 2x) \, dx = \boxed{-\frac{\cos 6x}{12} - \frac{\cos 2x}{4} + C}.$$

Math 125 Final, Autumn 2023 — Problem 2

(a) By parts, $u = \tan^{-1} x$, $dv = x \, dx$:

$$\left[\frac{x^2}{2} \tan^{-1} x \right]_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} dx = \frac{\pi}{8} - \frac{1}{2} \left(1 - \frac{\pi}{4} \right) = \boxed{\frac{\pi}{4} - \frac{1}{2}}.$$

(b) $2x^2 + 7x + 3 = (2x+1)(x+3)$ and $\frac{1}{(2x+1)(x+3)} = \frac{2/5}{2x+1} - \frac{1/5}{x+3}$:

$$\lim_{t \rightarrow \infty} \frac{1}{5} \left[\ln \frac{2x+1}{x+3} \right]_0^t = \frac{1}{5} \left(\ln 2 - \ln \frac{1}{3} \right) = \boxed{\frac{\ln 6}{5}} \approx 0.358.$$

Math 125 Final, Autumn 2023 — Problem 3

(a) By parts ($u = t - 1$, $dv = e^{3t}dt$):

$$s(t) = \int 9(t-1)e^{3t}dt = 3(t-1)e^{3t} - e^{3t} + C = (3t-4)e^{3t} + C.$$

$$s(0) = 0 \Rightarrow -4 + C = 0: \boxed{s(t) = (3t-4)e^{3t} + 4}.$$

(b) $v < 0$ on $(0, 1)$ and $v > 0$ on $(1, 2)$. $s(0) = 0$, $s(1) = 4 - e^3$, $s(2) = 2e^6 + 4$:

$$\text{distance} = |s(1) - s(0)| + |s(2) - s(1)| = (e^3 - 4) + (2e^6 + e^3) = \boxed{2e^6 + 2e^3 - 4} \approx 843.0 \text{ ft.}$$

Math 125 Final, Autumn 2023 — Problem 4

$$y' = \frac{2}{3} \cdot \frac{3}{2}(x^2 + 1)^{1/2}(2x) = 2x\sqrt{x^2 + 1}, \text{ so}$$

$$1 + (y')^2 = 1 + 4x^2(x^2 + 1) = 4x^4 + 4x^2 + 1 = (2x^2 + 1)^2.$$

$$L = \int_{-1}^1 (2x^2 + 1) \, dx = \left[\frac{2}{3}x^3 + x \right]_{-1}^1 = \boxed{\frac{10}{3}}.$$

Math 125 Final, Autumn 2023 — Problem 5

Put the tank's center at $y = 0$; the top of the tank is $y = 1$ and the spout outlet is $y = 2$. Half full means oil fills $-1 \leq y \leq 0$. A slice at height y has area $\pi(1 - y^2)$, weighs $60\pi(1 - y^2) dy$ lb, and is lifted $2 - y$ ft:

$$W = \int_{-1}^0 60\pi(1 - y^2)(2 - y) dy = 60\pi \left[2y - \frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4} \right]_{-1}^0 = 60\pi \cdot \frac{19}{12} = \boxed{95\pi \approx 298.5 \text{ ft-lb}}.$$

Math 125 Final, Autumn 2023 — Problem 6

(a) Total area $\int_0^2 x^4 dx = \frac{32}{5}$. Need $\int_0^a x^4 dx = \frac{a^5}{5} = \frac{16}{5}$, so $a = 16^{1/5} = \boxed{2^{4/5} \approx 1.741}$.

(b) SHELLS (horizontal shells, $0 \leq y \leq 16$, from $x = y^{1/4}$ to $x = 2$, radius $y + 3$):

$$V = \int_0^{16} 2\pi(y + 3)(2 - y^{1/4}) dy.$$

WASHERS (vertical slices, outer radius $x^4 + 3$, inner radius 3):

$$V = \int_0^2 \pi[(x^4 + 3)^2 - 3^2] dx.$$

Math 125 Final, Autumn 2023 — Problem 7

This is a quarter of the ellipse $\frac{x^2}{4} + y^2 = 1$ ($a = 2$, $b = 1$), area $\frac{\pi ab}{4} = \frac{\pi}{2}$.

$$M_y = \int_0^2 x \sqrt{1 - \frac{x^2}{4}} \, dx = \left[-\frac{4}{3} \left(1 - \frac{x^2}{4}\right)^{3/2} \right]_0^2 = \frac{4}{3}, \qquad M_x = \int_0^2 \frac{1}{2} \left(1 - \frac{x^2}{4}\right) dx = \frac{1}{2} \left(2 - \frac{2}{3}\right) = \frac{2}{3}.$$

$$\bar{x} = \frac{4/3}{\pi/2} = \frac{8}{3\pi}, \quad \bar{y} = \frac{2/3}{\pi/2} = \frac{4}{3\pi} : \quad \boxed{\left(\frac{8}{3\pi}, \frac{4}{3\pi}\right)} \approx (0.849, 0.424).$$

Math 125 Final, Autumn 2023 — Problem 8

$y^2 - 3y + 2 = (y - 1)(y - 2)$. Separate with partial fractions $\frac{1}{(y-1)(y-2)} = \frac{1}{y-2} - \frac{1}{y-1}$:

$$\ln \left| \frac{y-2}{y-1} \right| = \sin x + C.$$

$y(0) = 3$: $C = \ln \frac{1}{2}$. Near $y = 3$ the ratio is positive, so $\frac{y-2}{y-1} = \frac{1}{2}e^{\sin x}$. Solve for y :

$y = \frac{4 - e^{\sin x}}{2 - e^{\sin x}}$

(check: $y(0) = \frac{3}{1} = 3$).

Math 125 Final, Autumn 2023 — Problem 9

(a) $y = 10e^{-\alpha t}$ and $8 = 10e^{-3\alpha}$, so $\alpha = \frac{\ln(5/4)}{3} \approx 0.07438$. Then $2 = 10e^{-\alpha t}$ gives

$$t = \frac{\ln 5}{\alpha} = \frac{3 \ln 5}{\ln 1.25} \approx \boxed{21.64 \text{ days}}.$$

(b) $y' = -\alpha y - 0.5$ has equilibrium $-\frac{0.5}{\alpha}$, so $y = -\frac{0.5}{\alpha} + \left(12 + \frac{0.5}{\alpha}\right)e^{-\alpha t}$. With $c = \frac{0.5}{\alpha} \approx 6.722$, set $y = 2$:

$$e^{-\alpha t} = \frac{2+c}{12+c} \Rightarrow t = \frac{1}{\alpha} \ln \frac{12+c}{2+c} \approx \boxed{10.27 \text{ days}}.$$