

**Math 125 Final, Spring 2024 — Problem 1**

(a) Long division:  $\frac{2x^4 - 29x^2 + 12x + 1}{x^2 + 4x} = 2x^2 - 8x + 3 + \frac{1}{x^2 + 4x}$ , and  $\frac{1}{x(x+4)} = \frac{1}{4}\left(\frac{1}{x} - \frac{1}{x+4}\right)$ :

$$\boxed{\frac{2}{3}x^3 - 4x^2 + 3x + \frac{1}{4}\ln\left|\frac{x}{x+4}\right| + C}.$$

(b)  $x^2 + 6x + 1 = (x+3)^2 - 8$ . With  $x+3 = \sqrt{8}\sec\theta$ ,  $\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|$ :

$$\boxed{\ln\left|x+3+\sqrt{x^2+6x+1}\right|+C}.$$

**Math 125 Final, Spring 2024 — Problem 2**

(a) By parts twice:

$$\int x^2 \sin 3x \, dx = -\frac{x^2 \cos 3x}{3} + \frac{2x \sin 3x}{9} + \frac{2 \cos 3x}{27}.$$

From 0 to  $\pi$  ( $\cos 3\pi = -1$ ):  $\left(\frac{\pi^2}{3} - \frac{2}{27}\right) - \frac{2}{27} = \boxed{\frac{\pi^2}{3} - \frac{4}{27}} \approx 3.142$ .

(b) Let  $u = \sqrt{x}$ ,  $dx = 2u \, du$ :

$$\int_0^{\sqrt{3}} \frac{2u^2}{u^2 + 9} du = 2 \int_0^{\sqrt{3}} \left(1 - \frac{9}{u^2 + 9}\right) du = 2 \left[ u - 3 \tan^{-1} \frac{u}{3} \right]_0^{\sqrt{3}} = \boxed{2\sqrt{3} - \pi}.$$

Math 125 Final, Spring 2024 — Problem 3

From the description:  $f = 1$  on  $[0, 1]$ ; line to  $(2, 2)$ ; line from  $(2, 2)$  through  $(4, 0)$  to  $(5, -1)$ ; lower semicircle of radius 1 centered  $(6, -1)$  (bottom at  $(6, -2)$ ); line from  $(7, -1)$  to  $(8, 1)$ .

(a)  $\int_0^5 f = 1 + \frac{1+2}{2} + \underbrace{2}_{[2,4]} - \underbrace{\frac{1}{2}}_{[4,5]} = \boxed{4}$ .

(b)  $g'(x) = 3f(3x)$ , so  $g'(2) = 3f(6) = 3(-2) = \boxed{-6}$ .

(c)  $\int_6^7 f = -(1 + \frac{\pi}{4})$  (unit square plus a quarter disk, below the axis);  $\int_7^8 f = 0$  (the line goes from  $-1$  to  $1$ ).

Average:  $\frac{-(1 + \pi/4)}{2} \approx \boxed{-0.893}$ .

(d) The sum is a Riemann sum for  $\int_5^7 f(t) dt = -(2 + \frac{\pi}{2}) \approx \boxed{-3.571}$ .

**Math 125 Final, Spring 2024 — Problem 4**

Measure distance fallen  $d(t)$  from the top.  $d'' = 9.8$ , so  $d'(t) = v_0 + 9.8t$  and  $d'(2) = 20 \Rightarrow v_0 = 0.4$  m/s. Then  $d(t) = 0.4t + 4.9t^2$ .

The building height is  $d(5) = 2 + 122.5 = 124.5$  m. The window is  $d(2) = 0.8 + 19.6 = 20.4$  m below the top, so its height above the ground is

$$124.5 - 20.4 = \boxed{104.1 \text{ m}}.$$

**Math 125 Final, Spring 2024 — Problem 5**

Area of  $R$ :  $\int_0^8 x^{1/3} dx = \frac{3}{4} \cdot 8^{4/3} = 12$ . The line  $y = bx$  meets  $y = x^{1/3}$  at  $x = b^{-3/2}$ ; for the answer below that point is past  $x = 8$ , so on  $[0, 8]$  the line stays below the curve and the part of  $R$  under the line is the triangle  $\int_0^8 bx \, dx = 32b$ . Set it to half:

$$32b = 6 \Rightarrow \boxed{b = \frac{3}{16}}.$$

(Check: at  $x = 8$  the line is at  $1.5 < 2$ , and they meet at  $x = (16/3)^{3/2} \approx 12.3 > 8$ .)

**Math 125 Final, Spring 2024 — Problem 6**

The curves meet where  $4x - x^2 = x$ :  $x = 0$  and  $x = 3$ ; on  $[0, 3]$  the top curve is  $y = 4x - x^2$ .

(a) Shells about  $x = -1$  (radius  $x + 1$ , height  $3x - x^2$ ):

$$V = \int_0^3 2\pi(x + 1)(3x - x^2) dx.$$

(b) Washers about  $y = 4$  (both curves lie below  $y = 4$ ; the outer radius goes to the lower curve  $y = x$ ):

$$V = \int_0^3 \pi \left[ (4 - x)^2 - (4 - (4x - x^2))^2 \right] dx.$$

Math 125 Final, Spring 2024 — Problem 7

(a) The region’s height at  $x$  is  $\frac{2}{x^3}$ .

$$\bar{x} = \frac{\int_1^a x \cdot \frac{2}{x^3} dx}{\int_1^a \frac{2}{x^3} dx} = \frac{2(1 - \frac{1}{a})}{1 - \frac{1}{a^2}} = \frac{2}{1 + \frac{1}{a}} = \boxed{\frac{2a}{a + 1}}.$$

(b) As  $a \rightarrow \infty$ ,  $\bar{x} \rightarrow \boxed{2}$ : the centroid approaches  $(2, 0)$ .

**Math 125 Final, Spring 2024 — Problem 8**

Put  $y = 0$  at the cone's tip and pump to the top,  $y = 8$ . The radius at the top is 4.5 m.

Cone part,  $0 \leq y \leq 3$ : radius  $1.5y$ . Cylinder part,  $3 \leq y \leq 8$ : radius 4.5. Each slice is lifted  $8 - y$  m:

$$W = 9800\pi \left[ \int_0^3 (1.5y)^2(8 - y) dy + \int_3^8 (4.5)^2(8 - y) dy \right] = 9800\pi (116.4375 + 253.125) \approx \boxed{1.138 \times 10^7 \text{ J}}.$$

Math 125 Final, Spring 2024 — Problem 9

Separate:  $e^{-y}dy = \frac{e^x}{1 + e^x}dx \Rightarrow -e^{-y} = \ln(1+e^x)+C$ .  $y(0) = -2$ :  $-e^2 = \ln 2+C$ , so  $e^{-y} = e^2+\ln 2-\ln(1+e^x)$ :

$y = -\ln(e^2 + \ln 2 - \ln(1 + e^x))$ .

**Math 125 Final, Spring 2024 — Problem 10**

(a) Net outflow is 1.5 L/s:  $V(t) = 100 - 1.5t$ .

(b) Only salt leaves, at concentration  $\frac{y}{V}$ :

$$\frac{dy}{dt} = -\frac{3.5y}{100 - 1.5t}, \quad y(0) = 10.$$

(c) Separate:  $\ln y = \frac{3.5}{1.5} \ln(100 - 1.5t) + C$ , so  $y = 10 \left( \frac{100 - 1.5t}{100} \right)^{7/3}$ . At  $t = 30$ :

$$y = 10(0.55)^{7/3} \approx \boxed{2.478 \text{ g}}.$$