

Math 125 Final, Winter 2015 — Problem 1

(a) $1 + \tan^2 \theta = \sec^2 \theta$, so the integrand is $\sec^4 \theta = (1 + \tan^2 \theta) \sec^2 \theta$. With $u = \tan \theta$:

$$\int (1 + u^2) du = \boxed{\tan \theta + \frac{1}{3} \tan^3 \theta + C}.$$

(b) Complete the square: $x^2 + 4x + 13 = (x + 2)^2 + 9$. Let $u = x + 2$:

$$\int \frac{u - 2}{(u^2 + 9)^{3/2}} du = -\frac{1}{\sqrt{u^2 + 9}} - 2 \int \frac{du}{(u^2 + 9)^{3/2}}.$$

With $u = 3 \tan \theta$, $\int \frac{du}{(u^2 + 9)^{3/2}} = \frac{1}{9} \sin \theta = \frac{u}{9\sqrt{u^2 + 9}}$. So

$$\boxed{-\frac{1}{\sqrt{x^2 + 4x + 13}} - \frac{2(x + 2)}{9\sqrt{x^2 + 4x + 13}} + C}.$$

Math 125 Final, Winter 2015 — Problem 2

(a) Let $u = 4 - x$ ($x = 4 - u$, u from 1 to 0):

$$\int_0^1 (4 - u)\sqrt{u} \, du = \left[\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2} \right]_0^1 = \frac{8}{3} - \frac{2}{5} = \boxed{\frac{34}{15}}.$$

(b) $x^2 + 4x + 3 = (x + 1)(x + 3)$ and $\frac{4-2x}{(x+1)(x+3)} = \frac{3}{x+1} - \frac{5}{x+3}$:

$$\left[3 \ln(x + 1) - 5 \ln(x + 3) \right]_0^2 = 3 \ln 3 - 5 \ln 5 + 5 \ln 3 = \boxed{8 \ln 3 - 5 \ln 5} \approx 0.742.$$

Math 125 Final, Winter 2015 — Problem 3

(a) If $p \geq 0$ the integrand $xe^{px} \geq x$ for $x \geq 0$, so the integral diverges. If $p < 0$, $xe^{px} \rightarrow 0$ fast enough. It converges exactly for $\boxed{p < 0}$.

(b) For $p < 0$, by parts:

$$\int_0^t xe^{px} dx = \left[\frac{xe^{px}}{p} - \frac{e^{px}}{p^2} \right]_0^t \rightarrow 0 - \left(-\frac{1}{p^2} \right) = \boxed{\frac{1}{p^2}}.$$

Math 125 Final, Winter 2015 — Problem 4

Let the barriers be at $x = L(t)$ and $x = R(t)$. The area is $A = \int_L^R \sqrt{6 + \cos x} \, dx$. By the Fundamental Theorem (and the chain rule):

$$\frac{dA}{dt} = \sqrt{6 + \cos R} \frac{dR}{dt} - \sqrt{6 + \cos L} \frac{dL}{dt}.$$

At this instant $R = \pi$, $\frac{dR}{dt} = \beta$, $L = -\pi$, $\frac{dL}{dt} = -\alpha$, and $\sqrt{6 + \cos(\pm\pi)} = \sqrt{5}$:

$$\frac{dA}{dt} = \boxed{\sqrt{5} (\alpha + \beta)} \text{ units}^2/\text{sec}.$$

Math 125 Final, Winter 2015 — Problem 5

$v(t) = \int (2t - 3) dt = t^2 - 3t + C$, $v(0) = -4$: $v(t) = t^2 - 3t - 4 = (t - 4)(t + 1)$. It changes sign at $t = 4$, so we go to $t = 6$:

$$\int_0^4 (4 + 3t - t^2) dt + \int_4^6 (t^2 - 3t - 4) dt = \frac{56}{3} + \frac{38}{3} = \boxed{\frac{94}{3} \approx 31.33 \text{ m}}.$$

Math 125 Final, Winter 2015 — Problem 6

(a) At position x along the long axis, the rim's width is $2y = 2b\sqrt{1 - x^2/a^2}$, which is the side of the square cross-section. Its area is $4b^2(1 - x^2/a^2)$:

$$V = \int_{-a}^a 4b^2 \left(1 - \frac{x^2}{a^2}\right) dx.$$

(b)

$$V = 4b^2 \left[x - \frac{x^3}{3a^2} \right]_{-a}^a = 4b^2 \cdot \frac{4a}{3} = \boxed{\frac{16ab^2}{3} \text{ m}^3}.$$

Math 125 Final, Winter 2015 — Problem 7

$\Delta x = \frac{7-(-2)}{3} = 3$, nodes $-2, 1, 4, 7$. With $g(x) = \frac{1}{1+(f(x))^2}$:

$$T_3 = \frac{3}{2} \Big[g(-2) + 2g(1) + 2g(4) + g(7) \Big] = \frac{3}{2} \Big[\frac{1}{1+f(-2)^2} + \frac{2}{1+f(1)^2} + \frac{2}{1+f(4)^2} + \frac{1}{1+f(7)^2} \Big].$$

Math 125 Final, Winter 2015 — Problem 8

$$\bar{x} = \frac{\int_1^e x \cdot \frac{\ln x}{x^2} dx}{\int_1^e \frac{\ln x}{x^2} dx}.$$

$$\text{Numerator: } \int_1^e \frac{\ln x}{x} dx = \left[\frac{(\ln x)^2}{2} \right]_1^e = \frac{1}{2}.$$

$$\text{Denominator (by parts): } \int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x}, \text{ so } \left[\cdot \right]_1^e = -\frac{2}{e} + 1.$$

$$\bar{x} = \frac{1/2}{1 - 2/e} = \boxed{\frac{e}{2(e - 2)}} \approx 1.892.$$

Math 125 Final, Winter 2015 — Problem 9

Separate: $\frac{dy}{y+2\sqrt{y}} = \cos^2 t \, dt$. On the left, $y+2\sqrt{y} = \sqrt{y}(\sqrt{y}+2)$; let $u = \sqrt{y}$, $dy = 2u \, du$:

$$\int \frac{2u \, du}{u(u+2)} = 2 \ln(\sqrt{y}+2), \quad \int \cos^2 t \, dt = \frac{t}{2} + \frac{\sin 2t}{4}.$$

So $2 \ln(\sqrt{y}+2) = \frac{t}{2} + \frac{\sin 2t}{4} + C$, and $y(0) = 1$ gives $C = 2 \ln 3$. Then $\sqrt{y}+2 = 3e^{t/4+\sin(2t)/8}$:

$$y = \left(3e^{t/4+\sin(2t)/8} - 2 \right)^2.$$

Math 125 Final, Winter 2015 — Problem 10

(a) Salt in: $4 + 3(2) = 10$ g/min. Salt out: $2 \cdot \frac{y}{20} = \frac{y}{10}$ g/min (the volume stays at 20 L):

$$\boxed{\frac{dy}{dt} = 10 - \frac{y}{10}, \quad y(0) = 400}.$$

(b) Separate: $\frac{dy}{100 - y} = \frac{dt}{10} \Rightarrow -\ln|100 - y| = \frac{t}{10} + C \Rightarrow y = 100 + Ke^{-t/10}$. $y(0) = 400 \Rightarrow K = 300$:

$$\boxed{y(t) = 100 + 300e^{-t/10}}.$$

(c) $\lim_{t \rightarrow \infty} y(t) = \boxed{100 \text{ g}}$.