

Your Name

Your Signature

Student number

- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed books. You may use one $8\frac{1}{2}'' \times 11''$ sheet of handwritten notes (you can write on both sides).
- You can use only Texas Instruments TI-30X calculator.
- Give your answers in exact form, not decimals, unless instructed otherwise.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- **Check your work carefully.** We will award only limited partial credit.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 7 pages, plus this cover sheet. Make sure that your exam is complete.

1. (10 points) Compute the integral.

$$\int (4x^2 - 9)^{-5/2} dx$$

Solution:

We use substitution $x = (3/2) \sec \theta$, $dx = (3/2) \sec \theta \tan \theta d\theta$.

$$\begin{aligned} \int (4x^2 - 9)^{-5/2} dx &= \int \frac{1}{(\sqrt{4x^2 - 9})^5} dx = \int \frac{1}{(\sqrt{4(9/4) \sec^2 \theta - 9})^5} (3/2) \sec \theta \tan \theta d\theta \\ &= \frac{3}{2} \cdot \frac{1}{3^5} \int \frac{1}{(\sqrt{\sec^2 \theta - 1})^5} \sec \theta \tan \theta d\theta = \frac{3}{2} \cdot \frac{1}{3^5} \int \frac{1}{(\sqrt{\tan^2 \theta})^5} \sec \theta \tan \theta d\theta \\ &= \frac{1}{162} \int \frac{\sec \theta}{\tan^4 \theta} d\theta = \frac{1}{162} \int \frac{\cos^3 \theta}{\sin^4 \theta} d\theta = \frac{1}{162} \int \frac{(1 - \sin^2 \theta) \cos \theta}{\sin^4 \theta} d\theta. \end{aligned}$$

We use substitution $u = \sin \theta$, $du = \cos \theta d\theta$.

$$\begin{aligned} &= \frac{1}{162} \int \frac{(1 - \sin^2 \theta) \cos \theta}{\sin^4 \theta} d\theta = \frac{1}{162} \int \frac{1 - u^2}{u^4} du = \frac{1}{162} \int (u^{-4} - u^{-2}) du = \frac{1}{162} \left(-\frac{1}{3u^3} + \frac{1}{u} \right) + C \\ &= \frac{1}{162} \left(-\frac{1}{3 \sin^3 \theta} + \frac{1}{\sin \theta} \right) + C \end{aligned}$$

We have $\sec \theta = \frac{2x}{3}$ so $\cos \theta = \frac{3}{2x}$

$$\sec \theta = \frac{2x}{3}$$

$$\cos \theta = \frac{3}{2x}$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - 9/(4x^2)} = \frac{\sqrt{4x^2 - 9}}{2x}$$

so

$$\begin{aligned} \frac{1}{162} \left(-\frac{1}{3 \sin^3 \theta} + \frac{1}{\sin \theta} \right) + C &= \frac{1}{162} \left(\frac{2x}{\sqrt{4x^2 - 9}} - \frac{8x^3}{3(4x^2 - 9)^{3/2}} \right) + C \\ &= \boxed{\frac{1}{81} \left(\frac{x}{\sqrt{4x^2 - 9}} - \frac{4x^3}{3(4x^2 - 9)^{3/2}} \right) + C} \end{aligned}$$

2. (10 points) Compute the integral.

$$\int \frac{x^{20} + x^{18} + 1}{x^3 + x} dx$$

Solution:

$$\begin{aligned} \int \frac{x^{20} + x^{18} + 1}{x^3 + x} dx &= \int \frac{x^{17}(x^3 + x) + 1}{x^3 + x} dx = \int x^{17} dx + \int \frac{1}{x^3 + x} dx, \\ \int x^{17} dx &= \frac{1}{18}x^{18}. \end{aligned}$$

To compute $\int \frac{1}{x^3 + x} dx$, we represent the integrand using partial fractions,

$$\frac{1}{x^3 + x} = \frac{1}{x(x^2 + 1)} = \frac{1}{x} - \frac{x}{x^2 + 1}.$$

Hence

$$\int \frac{1}{x^3 + x} dx = \int \left(\frac{1}{x} - \frac{x}{x^2 + 1} \right) dx = \ln|x| - \int \frac{x}{x^2 + 1} dx$$

We use substitution $u = x^2 + 1$, $du/2 = x dx$,

$$\int \frac{x}{x^2 + 1} dx = \int \frac{1}{2} \frac{1}{u} du = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln(x^2 + 1) + C$$

We combine partial results:

$$\int \frac{x^{20} + x^{18} + 1}{x^3 + x} dx = \boxed{\frac{1}{18}x^{18} + \ln|x| - \frac{1}{2} \ln(x^2 + 1) + C}$$

3. (10 points) In a physics experiment, a non-moving particle was isolated. Then it was given the acceleration of $\sin^2 t$ m/sec² for $\pi/2$ seconds, where t is the time in seconds after the start of the motion. Find the average velocity of the particle during the first $\pi/2$ seconds.

Give your answer in the exact form or as a decimal number with 5 significant digits.

Solution:

Let $v(t)$ denote the velocity.

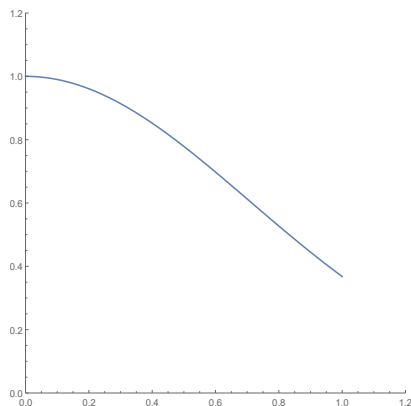
$$v(t) = \int_0^t \sin^2 s \, ds = \int_0^t \frac{1}{2}(1 - \cos(2s)) \, ds = \frac{1}{2} \left(s - \frac{1}{2} \sin(2s) \right) \Big|_0^t = \frac{1}{2}t - \frac{1}{4} \sin(2t).$$

The average velocity is

$$\begin{aligned} \frac{1}{\pi/2 - 0} \int_0^{\pi/2} v(t) \, dt &= \frac{1}{\pi/2 - 0} \int_0^{\pi/2} \left(\frac{1}{2}t - \frac{1}{4} \sin(2t) \right) \, dt = \frac{2}{\pi} \left(\frac{1}{4}t^2 + \frac{1}{8} \cos(2t) \right) \Big|_0^{\pi/2} \\ &= \frac{2}{\pi} \left(\frac{1}{4}(\pi/2)^2 - \frac{1}{8} - \frac{1}{8} \right) = \boxed{\frac{\pi}{8} - \frac{1}{2\pi}} \approx 0.233544 \end{aligned}$$

4. (10 points) The region A is below the curve $y = e^{-x^2}$, above the x -axis, between $x = 0$ and $x = 1$. Write an integral representing the volume of the body of revolution obtained by rotating the region about the x -axis. Apply the disc/washer/annulus method. Do not evaluate the integral. Instead, estimate the volume using Simpson's method with 4 subintervals.

Give your answer as a decimal number with 5 significant digits.



Solution: The volume is equal to

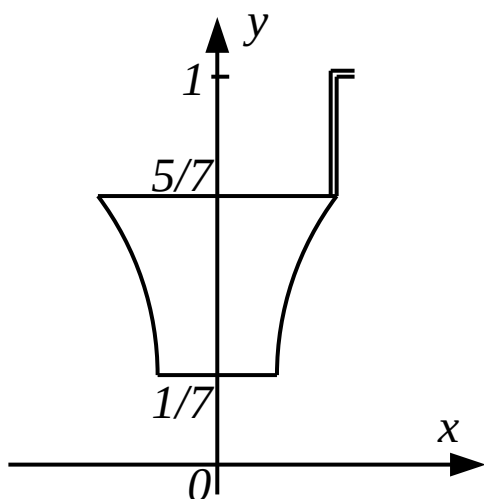
$$\int_0^1 \pi (e^{-x^2})^2 dx = \int_0^1 \pi e^{-2x^2} dx.$$

The Simpson method approximation gives

$$\frac{1}{4} \cdot \frac{1}{3} \pi \left(e^{-2 \cdot 0^2} + 4e^{-2 \cdot (1/4)^2} + 2e^{-2 \cdot (1/2)^2} + 4e^{-2 \cdot (3/4)^2} + e^{-2 \cdot 1^2} \right) \approx \boxed{1.87893}$$

5. A region D lies above $y = 1/7$, below $y = 5/7$, to the right of the y -axis, and to the left of the graph of $x = \frac{1}{\sqrt{y}(1-y)}$. A vessel is a body of revolution obtained by rotating the region D about the y -axis. The vessel is filled with water. The water needs to be pumped through the spout located 1 meter above the ground. Note that the bottom of the vessel is $1/7$ meters above the ground.

The dimensions are in meters. Recall that the mass density of water is $1,000 \text{ kg/m}^3$ and the gravitational acceleration is 9.8 m/s^2 .



- (a) (4 points) How much work is needed to pump out a single layer of water y meters above the ground, with thickness Δy , through the spout?

Solution: A layer has circular shape with the radius $r = x = \frac{1}{\sqrt{y}(1-y)}$. The work needed for a single layer is

$$1,000 \cdot 9.8(1-y)\pi r^2 \Delta y = 1,000 \cdot 9.8(1-y)\pi \frac{1}{y(1-y)^2} \Delta y = \boxed{1,000 \cdot 9.8\pi \frac{1}{y(1-y)} \Delta y}$$

- (b) (2 points) Write a formula for the work needed to pump all the water through the spout.

Solution:

$$\boxed{\int_{1/7}^{5/7} 1,000 \cdot 9.8\pi \frac{1}{y(1-y)} dy}$$

- (c) (4 points) Find the total work by evaluating the integral. Give the answer in the exact form or in the decimal form with 5 significant digits.

Solution: We will use the following partial fractions representation:

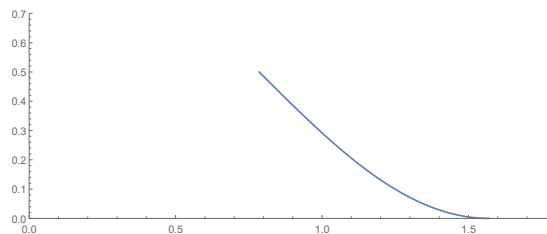
$$\frac{1}{y(1-y)} = \frac{1}{y} - \frac{1}{y-1}.$$

Hence

$$\begin{aligned} \int_{1/7}^{5/7} 1,000 \cdot 9.8\pi \frac{1}{y(1-y)} dy &= \int_{1/7}^{5/7} 9800\pi \left(\frac{1}{y} - \frac{1}{y-1} \right) dy = 9800\pi (\ln|y| - \ln|y-1|) \Big|_{1/7}^{5/7} \\ &= 9800\pi (\ln(5/7) - \ln(2/7) - \ln(1/7) + \ln(6/7)) = \boxed{9800\pi \ln(15)} \approx 83374.4(J) \end{aligned}$$

6. (10 points) A region A lies above the x -axis, below the graph of $y = \cos^2 x$, between $x = \pi/4$ and $x = \pi/2$ (see the figure below). Find the volume of the solid of revolution obtained by rotating the region A about the y -axis.

Give the answer in the exact form or in the decimal form with at least 5 significant digits.



Solution:

We will use the method of cylindrical shells. The volume is equal to

$$\int_{\pi/4}^{\pi/2} 2\pi x \cos^2 x dx.$$

To simplify the calculation, we will first compute the indefinite integral $\int x \cos^2 x dx$ by parts. If we take $u = x$ and $dv = \cos^2 x dx$ then $du = dx$ and v can be calculated as follows.

$$v = \int \cos^2 x dx = \int \frac{1}{2}(1 + \cos(2x)) dx = \frac{1}{2}(x + (1/2) \sin(2x)).$$

Hence

$$\begin{aligned} \int x \cos^2 x dx &= x \frac{1}{2}(x + (1/2) \sin(2x)) - \int \frac{1}{2}(x + (1/2) \sin(2x)) dx \\ &= x^2/2 + (1/4)x \sin(2x) - \frac{1}{2}((1/2)x^2 - (1/4) \cos(2x)) + C \\ &= x^2/4 + (1/4)x \sin(2x) + (1/8) \cos(2x) + C. \end{aligned}$$

The volume is equal to

$$\begin{aligned} \int_{\pi/4}^{\pi/2} 2\pi x \cos^2 x dx &= 2\pi \left(x^2/4 + (1/4)x \sin(2x) + (1/8) \cos(2x) \right) \Big|_{\pi/4}^{\pi/2} \\ &= 2\pi \left((\pi/2)^2/4 + (1/4)(\pi/2) \sin(2(\pi/2)) + (1/8) \cos(2(\pi/2)) \right) \\ &\quad - 2\pi \left((\pi/4)^2/4 + (1/4)(\pi/4) \sin(2(\pi/4)) + (1/8) \cos(2(\pi/4)) \right) \\ &= \boxed{\frac{3\pi^3 - 4\pi^2 - 8\pi}{32}} \approx 0.88774 \end{aligned}$$