

Math 125 A . Midterm 2. Fall 2024. Instructor: Elena Pezzoli.

NAME (First,Last) :

UW email:

Student ID:

- Unless the problem gives you specific instructions, you can give exact answers or approximate to two decimal digits.
- This exam is closed book. You may use one $8\frac{1}{2}'' \times 11''$ sheet of notes (both sides). Do not share notes.
- The only calculator allowed is a TI-30X IIS calculator.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- You may use any of the integral formulas in the table in the back of this page. All other integrals must be computed using the methods taught in this class and showing your work.
- Do not turn in any scratch work, or your note sheet. Only the work written on this exam will be graded. If you run out of space, continue your work on the back of the last page and indicate clearly on the problem page that you have done so.
- This exam will be scanned. Please make sure your writing is tidy and dark enough.
- This exam has 4 problems, on 4 pages, written both sides. The very last page is blank. Please make sure that your exam is complete.



Table of indefinite integrals.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1 ;$$

$$\int \frac{1}{x} dx = \ln |x| + C$$

$$\int e^x dx = e^x + C;$$

$$\int b^x dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x dx = -\cos x + C;$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C;$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C;$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \sec x dx = \ln |\sec x + \tan x| + C; \quad \int \csc x dx = \ln |\csc x - \cot x| + C$$

$$\int \tan x dx = \ln |\sec x| + C;$$

$$\int \cot x dx = \ln |\sin x| + C$$

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C;$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

Useful trig formulas:

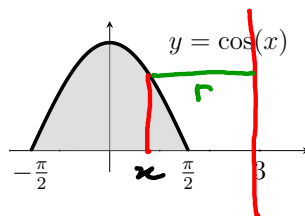
$$\sin^2 x + \cos^2 x = 1, \tan^2 x + 1 = \sec^2 x, \cos^2 x = \frac{1}{2}(1 + \cos(2x)), \sin^2 x = \frac{1}{2}(1 - \cos(2x)),$$

$$\sin(2x) = 2 \cos(x) \sin(x).$$

Trig special values :

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1

1. (10 points) The shaded region $\mathcal{R}$ below is bounded by the curves $y = \cos x$ and $y = 0$, between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.



- (a) Write an integral that computes the volume of the solid of revolution obtained by rotating $\mathcal{R}$ around the vertical line $x = 3$. Make sure you show all the steps you took to write this integral.

$$V_{\text{Solid}} = 2\pi r h \Delta x = 2\pi (3-x) \cos x \Delta x$$

In dy next page

ANSWER:

$$2\pi \int_{-\pi/2}^{\pi/2} (3-x) \cos x \, dx$$

- (b) Compute the integral you wrote in (a) (you will not get full credit for an integral computation if your integral from (a) is not correct).

$$\begin{aligned} 6\pi \int_{-\pi/2}^{\pi/2} \cos x \, dx - 2\pi \int_{-\pi/2}^{\pi/2} x \cos x \, dx &= 6\pi \sin x \Big|_{-\pi/2}^{\pi/2} - 2\pi x \sin x \Big|_{-\pi/2}^{\pi/2} \\ &+ 2\pi \int_{-\pi/2}^{\pi/2} \sin x \, dx = 6\pi \left(\sin \frac{\pi}{2} - \sin \left(-\frac{\pi}{2} \right) \right) - 2\pi \left(\frac{\pi}{2} \sin \frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \sin \left(-\frac{\pi}{2} \right) \right) \\ &+ \underbrace{2\pi (-\cos x) \Big|_{-\pi/2}^{\pi/2}}_{=0} \end{aligned}$$

ANSWER:

$$12\pi$$

Alternative solution

In Δy  $\Delta v = \pi (r_{out}^2 - r_{in}^2) \Delta y = \pi \left[\underbrace{(3 + \cos^{-1} y)^2}_{\geq 0} - \underbrace{(3 - \cos^{-1} y)^2}_{\geq 0} \right] \Delta y$

$$12\pi \int_0^1 \cos^{-1} y \, dy \quad \text{also ok}$$

$$12\pi \int_0^1 \underbrace{1}_{r'} \cdot \underbrace{\cos^{-1} y}_u \, dy = 12\pi \underbrace{y \cos^{-1} y}_0 \Big|_0^1 + 12\pi \int_0^1 \frac{y}{\sqrt{1-y^2}} \, dy$$

$$u = 1 - y^2$$

$$du = -2y \, dy$$

$$12\pi \int_0^1 \frac{1}{2\sqrt{u}} \, du$$

$$12\pi \sqrt{u} \Big|_0^1 = 12\pi$$

2. $f(x)$ is a function defined on the interval $[0,8]$. You do not have a formula for $f(x)$ but only a table of values. You know $f(x) \geq 0$, and $f(x)$ is concave up.

x	0	1	2	3	4	5	6	7	8
f(x)	4	0.8	0.2	0	0.2	0.4	1.8	1.6	25

- (a) (5 points) Give an underestimate of the area of the region $\mathcal{R}$ bounded by the curve $y = f(x)$ and the x axis between $x = 0$ and $x = 8$. Use $n = 4$. Show your work.



$$\Delta x = 2$$

choose x because the function is concave up

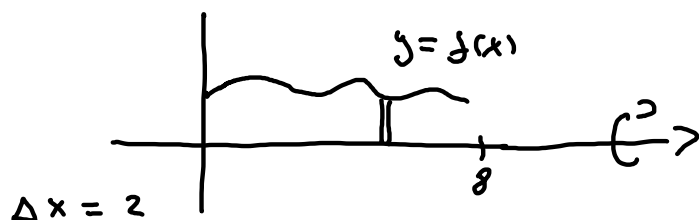
$$M_4: 2(f(1) + f(3) + f(5) + f(7))$$

$$= 2(4 + 0 + 0.4 + 1.6)$$

ANSWER:

48

- (b) (6 points) Use Simpson rule with $n = 4$ to approximate the volume of the solid of revolution obtained by rotating $\mathcal{R}$ around the x axis. Show your work. (Hint: first write an integral that calculates this volume).



$$V_{\text{slice}} = \pi r^2 \Delta x$$

$$\pi f(x)^2 \Delta x$$

$$\text{we get } \int_0^8 \pi f(x)^2 dx$$

$$S_4 = \frac{2}{3} \pi (f^2(0) + 4f^2(2) + 2f^2(4) + 4f^2(6) + f^2(8))$$

$$= \frac{2}{3} \pi (81 + 4 \cdot 1 + 2 \cdot 1 + 4 \cdot 81 + 625)$$

ANSWER:

$\frac{2072\pi}{3}$

1pt

3. Calculate the following integrals.

(a) (6 points) $\int_1^2 \frac{1}{x^2 - 5x} dx$

$$\frac{A}{x} + \frac{B}{x-5} = \frac{1}{x^2 - 5x} ; \quad A(x-5) + Bx = 1$$

$$\text{If } x = 0 \quad -5A = 1 \quad A = -\frac{1}{5} ; \quad \text{If } x = 5 \quad 5B = 1 \quad B = \frac{1}{5}$$

$$-\frac{1}{5} \int_1^2 \frac{1}{x} dx + \frac{1}{5} \int_1^2 \frac{1}{x-5} dx = -\frac{1}{5} \ln|x| \Big|_1^2 + \frac{1}{5} \ln|x-5| \Big|_1^2$$

$$= -\frac{1}{5} \ln 2 + \frac{1}{5} \ln 3 - \frac{1}{5} \ln 4 = \frac{1}{5} \left(\ln \frac{1}{2} + \ln 3 + \ln \frac{1}{4} \right)$$

ok answer

ANSWER:

$$\frac{1}{5} \ln\left(\frac{3}{8}\right)$$

(b) (6 points) $\int_0^{\pi/3} \sin^3(x) + \sec x \cdot \tan^3(x) dx$

$$\int_0^{\pi/3} \sin(x) \cdot \sin^2(x) dx + \int_0^{\pi/3} \sec(x) \tan x \tan^2(x) dx =$$

$$\int_0^{\pi/3} \sin x (1 - \cos^2 x) dx + \int_0^{\pi/3} \sec(x) \tan x (\sec^2 x - 1) dx$$

$$u = \cos x \\ du = -\sin x dx$$

$$u = \sec x \\ du = \sec x \tan x dx$$

$$-\int_1^{1/2} 1 - u^2 du + \int_1^2 u^2 - 1 du$$

$$u - \frac{u^3}{3} \Big|_{1/2}^1 + \left(\frac{u^3}{3} - u \right) \Big|_1^2 =$$

ANSWER (Give an exact answer simplified as much as possible):

$$\frac{37}{24}$$

$$\left(1 - \frac{1}{3}\right) - \left(\frac{1}{2} - \frac{1}{24}\right) + \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right) =$$

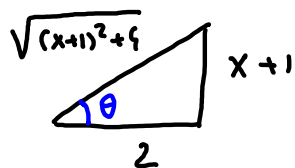
$$\frac{5}{24} + \frac{4}{3} = \frac{37}{24}$$

(c) (6 points) $\int \frac{1}{\sqrt{x^2 + 2x + 5}} dx$

$$\int \frac{1}{\sqrt{(x+1)^2 + 4}} dx$$

$$\begin{aligned} x+1 &= 2 \tan \theta \\ dx &= 2 \sec^2 \theta d\theta \end{aligned}$$

$$\int \frac{2 \sec^2 \theta d\theta}{\sqrt{4(\tan^2 \theta + 1)}} = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C$$



ANSWER:

$$\ln \left| \frac{\sqrt{(x+1)^2 + 4}}{2} + \frac{x+1}{2} \right| + C$$

4. This problem has two parts. In both parts a 10 feet chain is dangling from the roof of a tall building.

- (a) (4 points) The chain weighs 2 lb/ft. How much work is needed to pull it up to the top of the building?

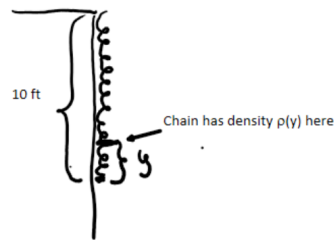
$$\int_0^{10} 2x \, dx = x^2 \Big|_0^{10} = 100$$

or $\int_0^{10} 2(10-x) \, dx$

ANSWER:

100 ft-lb

- (b) (6 points) The chain is made of different materials. Its weight density is not a uniform 2 lb/ft as in part (a) but it is given by $\rho(y) = \frac{y+10}{20}$ lb/ft where y is distance from the bottom of the chain. How much work is needed to pull the chain up to the top of the building?



$$w_{\text{slice}} = \underbrace{\rho(y) \Delta y}_{\text{weight}} \underbrace{(10-y)}_{\text{distance}}$$

$$\begin{aligned} \int_0^{10} \frac{y+10}{20} (10-y) \, dy &= \frac{1}{20} \int_0^{10} (100 - y^2) \, dy \\ &= \frac{1}{20} \left[100y - \frac{y^3}{3} \right]_0^{10} = \frac{1}{20} \left[1000 - \frac{1000}{3} \right] = \frac{2000}{30} \end{aligned}$$

ANSWER:

100/3 ft-lb