

Print Your Full Name

Solutions

Signature

Student ID Number

Quiz Section

Instructor's Name

TA's Name

Please read these instructions!

1. Your exam contains 7 pages; PLEASE MAKE SURE YOU HAVE A COMPLETE EXAM.
2. You are allowed a single, double-sided 8.5"x11" handwritten notesheet; the TI-30XIIS calculator; a writing utensil; and an eraser to use on the exam. No electronic devices are permitted during the exam.
3. The exam is worth 50 points. Point values for problems vary and these are clearly indicated. You have 80 minutes for this exam.
4. Unless otherwise stated, make sure to SHOW YOUR WORK CLEARLY. Credit is awarded to work which is clearly shown and legible. Full credit may not be awarded if work is unclear or illegible.
5. Unless otherwise indicated, please write your final answer in the provided location for each problem. Final answers do not need to be simplified, unless otherwise stated. If you need extra space, use the last two pages of the exam. Clearly indicate on that there is more work located on the last pages, and indicate on those pages the related problem number.
6. Unless otherwise instructed, always give your answers in exact form. For example, 3π , $\sqrt{2}$, and $\ln(2)$ are in exact form; the corresponding approximations 9.424778, 1.4142, and 0.693147 are NOT in exact form. Note that trigonometric functions on the unit circle, such as $\sin(\pi/6)$ and $\tan(\pi/3)$, must be written in fully simplified form (e.g., $\sin(\pi/6)$ is not simplified).
7. Credit is awarded for correct use of techniques or methods discussed in class thus far. Partial credit may be awarded as earned. No credit may be awarded for use of methods that are learned later in the course or from other (non-prerequisite) courses.

Table of Integration Formulas Constants of integration have been omitted.

- | | |
|---|---|
| 1. $\int x^n dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$ | 2. $\int \frac{1}{x} dx = \ln x $ |
| 3. $\int e^x dx = e^x$ | 4. $\int b^x dx = \frac{b^x}{\ln b}$ |
| 5. $\int \sin x dx = -\cos x$ | 6. $\int \cos x dx = \sin x$ |
| 7. $\int \sec^2 x dx = \tan x$ | 8. $\int \csc^2 x dx = -\cot x$ |
| 9. $\int \sec x \tan x dx = \sec x$ | 10. $\int \csc x \cot x dx = -\csc x$ |
| 11. $\int \sec x dx = \ln \sec x + \tan x $ | 12. $\int \csc x dx = \ln \csc x - \cot x $ |
| 13. $\int \tan x dx = \ln \sec x $ | 14. $\int \cot x dx = \ln \sin x $ |
| 17. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$ | 18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a > 0$ |

Problem	Total Points
1	10
2	8
3	8
4	12
5	12
Total	50

1. Evaluate the following integrals. Do not simplify your final answer.

(a) (5 points) $\int \frac{1}{(\sqrt{x})^5 + (\sqrt{x})^3} dx$

$u = \sqrt{x}$
 $du = \frac{1}{2\sqrt{x}} dx \rightarrow 2\sqrt{x} du = dx$
 $2u du = dx$

PFD: $\frac{1}{u^2(u^2+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C+D}{u^2+1}$

$1 = Au(u^2+1) + B(u^2+1) + (C+D)(u^2)$

$1 = u^3(A+C) + u^2(B+D) + u(A) + B$

$\Rightarrow A+C=0 \rightarrow C=0$
 $B+D=0 \rightarrow D=-1$
 $A=0$
 $B=1$

$= \int \frac{2u}{u^5 + u^3} du$

$= 2 \int \frac{1}{u^4 + u^2} du$

$= 2 \int \frac{1}{u^2(u^2+1)} du$

$= 2 \int \left(\frac{1}{u^2} - \frac{1}{u^2+1} \right) du$

$= 2 \left[\frac{u^{-1}}{-1} - \tan^{-1}(u) \right] + C = \boxed{2 \left[-\frac{1}{\sqrt{x}} - \tan^{-1}(\sqrt{x}) \right] + C}$

(b) (5 points) $\int \frac{(3 \sin \theta + 1)^2}{\cos^2 \theta} d\theta$

$= \int \frac{9 \sin^2 \theta + 6 \sin \theta + 1}{\cos^2 \theta} d\theta$

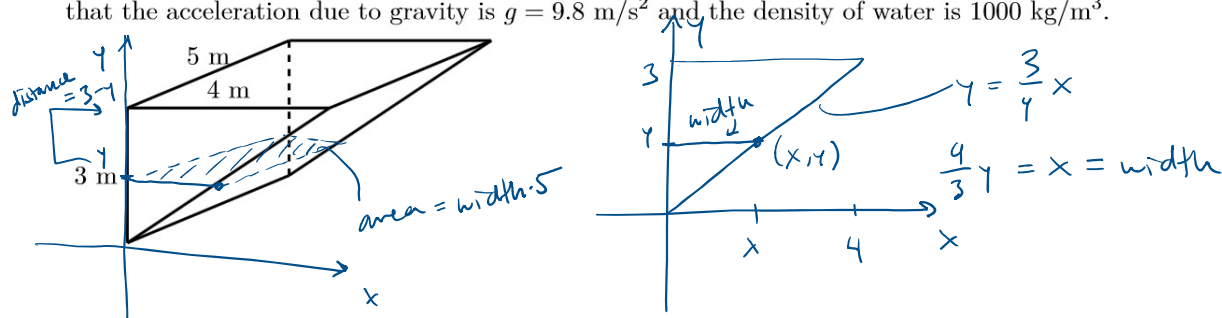
$= \int 9 \tan^2 \theta d\theta + 6 \int \frac{\sin \theta}{\cos^2 \theta} d\theta + \int \sec^2 \theta d\theta$

$= 9 \int (\sec^2 \theta - 1) d\theta + 6 \int -\frac{1}{u^2} du + \tan \theta$

$= 9 \left[\tan \theta - \theta \right] - 6 \frac{u^{-1}}{-1} + \tan \theta + C$

$= \boxed{9 \left[\tan \theta - \theta \right] + 6 \cdot \frac{1}{\cos \theta} + \tan \theta + C}$

2. (8 points) A triangular tank of height 3 meters, width 4 meters, and depth 5 meters is full of water (see the figure below). Assume the front and the back of the tank are triangles of the same dimensions. Set up, but do not evaluate, an integral equal to the work required to lift all the water to the top of the tank. Recall that the acceleration due to gravity is $g = 9.8 \text{ m/s}^2$ and the density of water is 1000 kg/m^3 .



$$W = \int_0^3 1000 \cdot 9.8 \cdot \left(\frac{4}{3} y \cdot 5 \right) \cdot (3-y) dy$$

3. (8 points) Use Simpson's Rule with $n = 4$ to approximate the average value of the function $f(x) = \frac{\sin x}{x}$ on the interval $[1, 3]$. Round your answer to three decimal places.

$$f_{\text{avg}} = \frac{1}{3-1} \int_1^3 \frac{\sin x}{x} dx$$

$$\Delta x = \frac{3-1}{4} = \frac{1}{2}$$



$$\approx \frac{1}{2} \cdot \frac{\Delta x}{3} \left(f(1) + 4f\left(\frac{3}{2}\right) + 2f\left(2\right) + 4f\left(\frac{5}{2}\right) + f(3) \right)$$

$$\approx \boxed{0.451}$$

4. (12 points) Evaluate the following improper integral. You must use limits and show your work to receive full credit.

$$\int_1^{\infty} \frac{1}{x^3 + x} dx$$

$$= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x(x^2+1)} dx$$

$$= \lim_{t \rightarrow \infty} \left[\int_1^t \left(\frac{1}{x} - \frac{x}{x^2+1} \right) dx \right]$$

$$= \lim_{t \rightarrow \infty} \left[\int_1^t \frac{1}{x} dx - \int_1^t \frac{x}{x^2+1} dx \right]$$

$$= \lim_{t \rightarrow \infty} \left[\int_1^t \frac{1}{x} dx - \frac{1}{2} \int_2^{t^2+1} \frac{1}{u} du \right]$$

$$= \lim_{t \rightarrow \infty} \left[\left[\ln|x| \right]_1^t - \frac{1}{2} \left[\ln|u| \right]_2^{t^2+1} \right]$$

$$= \lim_{t \rightarrow \infty} \left[\ln t - \underbrace{\ln(1)}_{=0} - \frac{1}{2} \left(\ln(t^2+1) - \ln(2) \right) \right]$$

$$= \lim_{t \rightarrow \infty} \left[\ln t - \frac{1}{2} \ln(t^2+1) + \frac{1}{2} \ln(2) \right]$$

$$= \lim_{t \rightarrow \infty} \left[\ln t - \ln \left((t^2+1)^{\frac{1}{2}} \right) + \frac{1}{2} \ln(2) \right]$$

$$= \lim_{t \rightarrow \infty} \left[\ln \left(\frac{t}{\sqrt{t^2+1}} \right) + \frac{1}{2} \ln(2) \right] = \overbrace{\ln(1)}^{=0} + \frac{1}{2} \ln(2)$$

$$= \boxed{\frac{1}{2} \ln(2)}$$

PFD:

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + (Bx+C)x$$

$$1 = x^2(A+B) + x(C) + A$$

$$\Rightarrow A+B=0 \rightarrow B=-1$$

$$C=0$$

$$A=1$$

$$u=x^2+1$$

$$x=1 \rightarrow u=2$$

$$du=2x dx$$

$$x=t \rightarrow u=t^2+1$$

$$\frac{1}{2} du = x dx$$

5. (12 points) Evaluate the following integral. Fully simplify any trigonometric expressions in your final answer. You do not need to simplify any other algebraic expressions in your final answer.

$$\begin{aligned} & \int \sqrt{-4x^2 - 4x + 1} dx \\ &= \int \sqrt{2 - 4\left(x + \frac{1}{2}\right)^2} dx \\ &= \int \sqrt{4\left(\frac{1}{2} - \left(x + \frac{1}{2}\right)^2\right)} dx \\ &= 2 \int \sqrt{\frac{1}{2} - \left(x + \frac{1}{2}\right)^2} dx \end{aligned}$$

$$= 2 \int \sqrt{\frac{1}{2} - \frac{1}{2} \sin^2 \theta} \cdot \sqrt{\frac{1}{2}} \cos \theta d\theta$$

$$= 2 \int \sqrt{\frac{1}{2}} \cdot \underbrace{\sqrt{1 - \sin^2 \theta}}_{=\cos^2 \theta} \cdot \sqrt{\frac{1}{2}} \cdot \cos \theta d\theta$$

$$= \int \cos^2 \theta d\theta \quad \left. \begin{array}{l} \text{double-angle} \\ \text{formula} \end{array} \right\}$$

$$= \int \frac{1}{2} (1 + \cos(2\theta)) d\theta$$

$$= \frac{1}{2} \theta + \frac{\sin(2\theta)}{4} + C \quad \left. \begin{array}{l} \text{using} \\ \sin(2\theta) \\ = 2 \sin \theta \cos \theta \end{array} \right\}$$

$$= \frac{1}{2} \theta + \frac{\sin \theta \cdot \cos \theta}{2} + C$$

$$= \boxed{\frac{1}{2} \sin^{-1}\left(\frac{x + \frac{1}{2}}{\sqrt{\frac{1}{2}}}\right) + \frac{1}{2} \cdot \frac{x + \frac{1}{2}}{\sqrt{\frac{1}{2}}} \cdot \frac{\sqrt{\frac{1}{2} - \left(x + \frac{1}{2}\right)^2}}{\sqrt{\frac{1}{2}}} + C}$$

Complete the square: $-4(x^2 + x) + 1$

$$= -4\left(x^2 + x + \frac{1}{4} - \frac{1}{4}\right) + 1$$

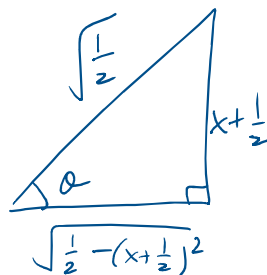
$$= -4\left(\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}\right) + 1$$

$$= -4\left(x + \frac{1}{2}\right)^2 + 2$$

Trig sub: $x + \frac{1}{2} = \sqrt{\frac{1}{2}} \cdot \sin \theta$

$$x = \sqrt{\frac{1}{2}} \cdot \sin \theta - \frac{1}{2}$$

$$dx = \sqrt{\frac{1}{2}} \cos \theta d\theta$$



$$\sin \theta = \frac{x + \frac{1}{2}}{\sqrt{\frac{1}{2}}}$$

$$\cos \theta = \frac{\sqrt{\frac{1}{2} - \left(x + \frac{1}{2}\right)^2}}{\sqrt{\frac{1}{2}}}$$

Extra scratch paper.

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