

Math 126 Final, Spring 2006 — Problem 1

(a) $f(x) = e^{-3}e^{2x}$ and $f^{(n)}(0) = 2^n e^{-3}$:

$$\boxed{T_3(x) = e^{-3}\left(1 + 2x + 2x^2 + \frac{4}{3}x^3\right)}.$$

(b) Taylor's inequality: $|f(x) - T_3(x)| \leq \frac{M}{4!}|x|^4$ where $M \geq |f^{(4)}| = 16e^{2x-3}$ on the interval. For $|x| < 0.5$, $16e^{2x-3} < 16e^{-2}$, so

$$|f(x) - T_3(x)| \leq \frac{16e^{-2}}{24}(0.5)^4 = \boxed{\frac{e^{-2}}{24} \approx 0.0056}.$$

Math 126 Final, Spring 2006 — Problem 2

(a) $\ln(2 + 5x) = \ln 2 + \ln\left(1 + \frac{5x}{2}\right)$ and $\ln(1 + u) = \sum_{n \geq 1} \frac{(-1)^{n+1} u^n}{n}$:

$$\boxed{\ln(2 + 5x) = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5^n}{n 2^n} x^n = \ln 2 + \frac{5}{2}x - \frac{25}{8}x^2 + \cdots}$$

(b) Ratio test: $\left| \frac{a_{n+1}}{a_n} \right| = \frac{n}{n+1} \cdot \frac{5|x|}{2} \rightarrow \frac{5|x|}{2}$, so it converges for $|x| < \frac{2}{5}$. At $x = \frac{2}{5}$ it is the alternating harmonic series (converges); at $x = -\frac{2}{5}$ it is $-\sum \frac{1}{n}$ (diverges). So it converges on $\boxed{\left(-\frac{2}{5}, \frac{2}{5}\right]}$ (any interval inside this, such as $(-0.4, 0.4)$, is also acceptable).

Math 126 Final, Spring 2006 — Problem 3

AD and BC join opposite corners of the cube, so they are main (body) diagonals. Put the cube at $[0, 1]^3$; two body diagonals are $\langle 1, 1, 1 \rangle$ and $\langle 1, 1, -1 \rangle$ (up to sign):

$$\cos \theta = \frac{|1 + 1 - 1|}{\sqrt{3}\sqrt{3}} = \frac{1}{3} \Rightarrow \theta = \cos^{-1} \frac{1}{3} \approx \boxed{70.5^\circ}.$$

Math 126 Final, Spring 2006 — Problem 4

Line 1: point $(0, -1, 5)$, direction $\langle 4, 2, -4 \rangle$. Line 2: set $x - 1 = 1 - y = \frac{z}{3} = s$: point $(1, 1, 0)$, direction $\langle 1, -1, 3 \rangle$. Normal:

$$\langle 4, 2, -4 \rangle \times \langle 1, -1, 3 \rangle = \langle 2, -16, -6 \rangle \parallel \langle 1, -8, -3 \rangle.$$

Through $(0, -1, 5)$: $x - 8(y + 1) - 3(z - 5) = 0$:

$$\boxed{x - 8y - 3z + 7 = 0} \quad (\text{check } (1, 1, 0) : 1 - 8 + 7 = 0).$$

Math 126 Final, Spring 2006 — Problem 5

For $y = x^4$: $y' = 4x^3$, $y'' = 12x^2$, so

$$\kappa(x) = \frac{12x^2}{(1 + 16x^6)^{3/2}}.$$

By the quotient rule, the numerator of $\kappa'(x)$ is $24x(1 + 16x^6)^{3/2} - 12x^2 \cdot \frac{3}{2}(1 + 16x^6)^{1/2} \cdot 96x^5$. Dividing by $(1 + 16x^6)^{1/2}$, $\kappa' = 0$ when

$$24x(1 + 16x^6) - 1728x^7 = 0 \Rightarrow 24x = 1344x^7 \Rightarrow x^6 = \frac{1}{56} \ (x \neq 0).$$

$\kappa(0) = 0$ and $\kappa \rightarrow 0$ as $|x| \rightarrow \infty$, so these give the maximum:

$$\boxed{\left(\pm 56^{-1/6}, 56^{-2/3} \right)} \approx (\pm 0.511, 0.068).$$

Math 126 Final, Spring 2006 — Problem 6

(a) $\vec{r}'(t) = \langle \pi \cos \pi t, -\pi \sin \pi t, 2t - 1 \rangle$, so $\vec{r}'(2) = \langle \pi, 0, 3 \rangle$ and $\vec{r}(2) = \langle 0, 1, 2 \rangle$. Normal plane:

$$\pi(x - 0) + 0(y - 1) + 3(z - 2) = 0 \Rightarrow \boxed{\pi x + 3z = 6}.$$

(b) $z = t^2 - t = 0$ at $t = 0$ and $t = 1$:

$$L = \int_0^1 \sqrt{\pi^2 \cos^2 \pi t + \pi^2 \sin^2 \pi t + (2t - 1)^2} dt = \int_0^1 \sqrt{\pi^2 + (2t - 1)^2} dt.$$

Math 126 Final, Spring 2006 — Problem 7

$$\vec{v} = \langle 2t, -t^{-2}, 1 \rangle, \vec{a} = \langle 2, 2t^{-3}, 0 \rangle.$$

(a) $\vec{v} \cdot \vec{a} = 4t - 2t^{-5} = 0 \Rightarrow t^6 = \frac{1}{2} \Rightarrow t = \boxed{2^{-1/6} \approx 0.891}.$

(b) There $\vec{a} \perp \vec{v}$, so $a_T = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|} = \boxed{0}$ and $a_N = |\vec{a}| = \sqrt{4 + 4t^{-6}} = \sqrt{4 + 8} = \boxed{2\sqrt{3}}.$

Math 126 Final, Spring 2006 — Problem 8

$x = r \cos \theta$, so $dx = \cos \theta \, dr - r \sin \theta \, d\theta$. At $r = 2$, $\theta = \frac{\pi}{6}$: $x = \sqrt{3}$ and

$$|dx| \leq \frac{\sqrt{3}}{2}(0.05) + 2 \cdot \frac{1}{2}(0.1) \approx 0.0433 + 0.1 = 0.1433.$$

So $x \approx \sqrt{3} \pm 0.143$, i.e. roughly $\boxed{1.589 \leq x \leq 1.875}$.

Math 126 Final, Spring 2006 — Problem 9

Minimize $M = x^2 + y^2 + z^2$ subject to $g = xy^2z^4 = 1$ (so $x, y, z > 0$). Lagrange: $\nabla M = \lambda \nabla g$:

$$2x = \lambda y^2 z^4, \quad 2y = 2\lambda xy z^4, \quad 2z = 4\lambda xy^2 z^3.$$

Multiply each by x, y, z and use $xy^2z^4 = 1$: $2x^2 = \lambda$, $2y^2 = 2\lambda$, $2z^2 = 4\lambda$, so $x^2 = \frac{\lambda}{2}$, $y^2 = \lambda$, $z^2 = 2\lambda$. Then $x^2 y^4 z^8 = \frac{\lambda}{2} \cdot \lambda^2 \cdot 16\lambda^4 = 8\lambda^7 = 1$, so $\lambda = 2^{-3/7}$:

$x = \sqrt{\lambda/2} \approx 0.610, \quad y = \sqrt{\lambda} \approx 0.862, \quad z = \sqrt{2\lambda} \approx 1.219$

with minimum $M = \frac{7}{2}\lambda \approx 2.600$.

Math 126 Final, Spring 2006 — Problem 10

(a) The solid lies over the half-disk where $z = x \geq 0$, under $z = x$. In polar coordinates:

$$V = \int_{-\pi/2}^{\pi/2} \int_0^2 (r \cos \theta) r \, dr \, d\theta \quad \left(\text{or } \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x \, dy \, dx \right).$$

(b) Reverse the order: the region is $0 \leq y \leq x \leq \frac{\pi}{2}$:

$$\int_0^{\pi/2} \int_0^x \frac{\sin x}{x} \, dy \, dx = \int_0^{\pi/2} \sin x \, dx = \boxed{1}.$$