

Math 126 Final, Spring 2007 — Problem 1

(a) $f(1) = 2$, $f'(x) = 3x^2 + 1 \Rightarrow f'(1) = 4$, $f''(x) = 6x \Rightarrow f''(1) = 6$:

$$T_2(x) = 2 + 4(x - 1) + 3(x - 1)^2.$$

(b) $f'''(x) = 6$ everywhere, so with $M = 6$: $|T_2(x) - f(x)| \leq \frac{6}{3!}|x - 1|^3 = |x - 1|^3$. This is less than 0.001 when $|x - 1| < 0.1$:

$$J = (0.9, 1.1).$$

Math 126 Final, Spring 2007 — Problem 2

$\frac{e^{x^2} - 1}{x} = \frac{1}{x} \left(x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots \right) = x + \frac{x^3}{2} + \cdots$, and $\frac{3}{(1-x)^2} = 3 \sum_{n \geq 0} (n+1)x^n = 3 + 6x + 9x^2 + 12x^3 + \cdots$.

Adding:

$f(x) = 3 + 7x + 9x^2 + \frac{25}{2}x^3 + \cdots$

Math 126 Final, Spring 2007 — Problem 3

- (a) **F**: $\vec{v} \times \vec{v} = \vec{0}$ for every $\vec{v}$. (b) **T**: $\vec{v} \cdot \vec{v} = |\vec{v}|^2$. (c) **F**: $\vec{v} \cdot \vec{w}$ is a scalar; you can't cross a vector with a scalar. (d) **T**: the triple product is a number. (e) **F**: $\vec{v} \cdot \vec{w}$ is a scalar, not a vector. (f) **T**. (g) **T**: the cross terms $-\vec{w} \cdot \vec{v} + \vec{v} \cdot \vec{w}$ cancel. (h) **F**: $(\vec{v} - \vec{w}) \times (\vec{v} + \vec{w}) = 2\vec{v} \times \vec{w}$, while $\vec{v} \times \vec{v} - \vec{w} \times \vec{w} = \vec{0}$.

Math 126 Final, Spring 2007 — Problem 4

(a) Direction $= \vec{n}_1 \times \vec{n}_2 = \langle 2, -1, 1 \rangle \times \langle 1, -1, 1 \rangle = \langle 0, -1, -1 \rangle$. Subtracting the equations gives $x = -1$; then $-y + z = 3$, e.g. $(-1, 0, 3)$:

$$\boxed{x = -1, \quad y = -t, \quad z = 3 - t}.$$

(b) Vectors in the plane: $\langle 0, -1, -1 \rangle$ and $(1, 1, -1) - (-1, 0, 3) = \langle 2, 1, -4 \rangle$. Normal $\langle 0, -1, -1 \rangle \times \langle 2, 1, -4 \rangle = \langle 5, -2, 2 \rangle$:

$$5(x - 1) - 2(y - 1) + 2(z + 1) = 0 \Rightarrow \boxed{5x - 2y + 2z = 1}.$$

Math 126 Final, Spring 2007 — Problem 5

$y = \ln \cos x$: $y' = -\tan x$, $y'' = -\sec^2 x$.

$$\kappa = \frac{|y''|}{(1 + (y')^2)^{3/2}} = \frac{\sec^2 x}{(\sec^2 x)^{3/2}} = |\cos x|.$$

The maximum curvature is $\boxed{1}$, at $x = 0$ (and $x = 2\pi k$).

Math 126 Final, Spring 2007 — Problem 6

$$\vec{v} = \langle 1, -t^{-2}, 2t \rangle, \vec{a} = \langle 0, 2t^{-3}, 2 \rangle.$$

(a) $\vec{v} \cdot \vec{a} = -2t^{-5} + 4t = 0 \Rightarrow t^6 = \frac{1}{2} \Rightarrow \boxed{t = 2^{-1/6} \approx 0.891}.$

(b) $a_T = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|} = \boxed{0}$ at that time.

Math 126 Final, Spring 2007 — Problem 7

(a) $y = 2$: $t^2 + t - 2 = 0 \Rightarrow t = 1$ or $t = -2$; $x(1) = 2$ ✓, $x(-2) = 8$. At $t = 1$:

$$\frac{dy}{dx} = \frac{2t+1}{4t^3+3t^2} = \frac{3}{7}, \quad \boxed{y-2 = \frac{3}{7}(x-2)}.$$

(b) Horizontal: $\frac{dy}{dt} = 2t+1 = 0$, $t = -\frac{1}{2}$ (there $\frac{dx}{dt} = \frac{1}{4} \neq 0$). Vertical: $\frac{dx}{dt} = t^2(4t+3) = 0$, $t = 0$ or $t = -\frac{3}{4}$ (there $\frac{dy}{dt} = 1$ and $-\frac{1}{2}$, both $\neq 0$).

Horizontal tangent at $\boxed{t = -\frac{1}{2}}$; vertical tangents at $\boxed{t = 0, -\frac{3}{4}}$.

Math 126 Final, Spring 2007 — Problem 8

(a) Need $2x - y > 0$ and $\ln(2x - y) \neq 0$: the domain is $\{(x, y) : y < 2x, y \neq 2x - 1\}$, the half-plane below the line $y = 2x$ with the line $y = 2x - 1$ removed.

(b) At (e, e) : $2x - y = e$, $\ln e = 1$, $f = 3e^2$. Quotient rule:

$$f_x = \frac{4x \ln(2x - y) - (2x^2 + y^2) \frac{2}{2x - y}}{\ln^2(2x - y)} = 4e - 6e = -2e, \quad f_y = \frac{2y \ln(2x - y) + (2x^2 + y^2) \frac{1}{2x - y}}{\ln^2(2x - y)} = 2e + 3e = 5e.$$

$$\boxed{z = 3e^2 - 2e(x - e) + 5e(y - e)}.$$

(c) $f(3, 3) \approx 3e^2 - 2e(3 - e) + 5e(3 - e) = 3e^2 + 3e(3 - e) = \boxed{9e \approx 24.46}$ (actual $\frac{27}{\ln 3} \approx 24.58$).

Math 126 Final, Spring 2007 — Problem 9

Let the base be x by y and the depth h , with the glass side being an x -by- h wall. Cost:

$$C = 15xy + \underbrace{15xh + 2 \cdot 15yh}_{\text{concrete sides}} + \underbrace{100xh}_{\text{glass}} = 15xy + 115xh + 30yh, \qquad xyh = 1000.$$

Substitute $h = \frac{1000}{xy}$: $C = 15xy + \frac{115000}{y} + \frac{30000}{x}$.

$$C_x = 15y - \frac{30000}{x^2} = 0 \Rightarrow y = \frac{2000}{x^2}, \qquad C_y = 15x - \frac{115000}{y^2} = 0.$$

Substituting: $x^3 = \frac{12 \times 10^6}{23000} \approx 521.7$, so

$$x \approx 8.05 \text{ m (glass wall length), } \quad y \approx 30.86 \text{ m, } \quad h \approx 4.03 \text{ m}$$

(cost $\approx$ \$11,180).

Math 126 Final, Spring 2007 — Problem 10

(a) $r = 2 \cos \theta$ is the circle of radius 1 centered at $(1, 0)$; $r = 2(1 + \cos \theta)$ is a cardioid through the origin, reaching $(4, 0)$. Since $2 \cos \theta \leq 2 + 2 \cos \theta$, the circle lies inside the cardioid; R is the region inside the cardioid and outside the circle.

(b)

$$A = \int_0^{2\pi} \int_0^{2(1+\cos\theta)} r \, dr \, d\theta - \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r \, dr \, d\theta = \int_0^{2\pi} 2(1+\cos\theta)^2 d\theta - \int_{-\pi/2}^{\pi/2} 2\cos^2\theta \, d\theta = 6\pi - \pi = \boxed{5\pi}.$$