

Math 126 Final, Spring 2008 — Problem 1

(a) $f(4) = 0 + 1 + 8 = 9$. $f'(x) = \cos(x - 4) - \sin(x - 4) + \frac{2}{\sqrt{x}}$, $f'(4) = 1 + 1 = 2$. $f''(x) = -\sin(x - 4) - \cos(x - 4) - x^{-3/2}$, $f''(4) = -1 - \frac{1}{8} = -\frac{9}{8}$.

$$T_2(x) = 9 + 2(x - 4) - \frac{9}{16}(x - 4)^2.$$

(b) $T_2(4.1) = 9 + 0.2 - \frac{9}{16}(0.01) = \boxed{9.194375}$.

(c) $f'''(x) = \sin(x-4)-\cos(x-4)+\frac{3}{2}x^{-5/2}$. On $[4, 4.1]$, $|\sin(x-4)-\cos(x-4)| \leq 1$ and $\frac{3}{2}x^{-5/2} \leq \frac{3}{2 \cdot 32} = 0.046875$, so take $M = 1.046875$:

$$|f(4.1) - T_2(4.1)| \leq \frac{M}{3!}(0.1)^3 \approx \boxed{1.75 \times 10^{-4}}.$$

Math 126 Final, Spring 2008 — Problem 2

(a) $\cos u = \sum \frac{(-1)^n u^{2n}}{(2n)!}$ with $u = x^2$: $1 - \cos(x^2) = \sum_{n \geq 1} \frac{(-1)^{n+1} x^{4n}}{(2n)!}$. Divide by x^3 :

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n-3}}{(2n)!}$$

(it gives $f(0) = 0$, matching the definition).

(b) $\frac{x}{2} - \frac{x^5}{24} + \frac{x^9}{720}$.

(c) The cosine series converges for every real number, so this series converges on $(-\infty, \infty)$.

Math 126 Final, Spring 2008 — Problem 3

(a) $\overrightarrow{PQ} = \langle 2, -1, 3 \rangle$, $\overrightarrow{PR} = \langle -1, 1, -1 \rangle$, normal $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -2, -1, 1 \rangle$:

$$-2(x - 1) - (y - 0) + (z - 2) = 0 \Rightarrow \boxed{2x + y - z = 0}.$$

(b) $\text{proj}_{\overrightarrow{PR}} \overrightarrow{PQ} = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{|\overrightarrow{PR}|^2} \overrightarrow{PR} = \frac{-6}{3} \langle -1, 1, -1 \rangle = \langle 2, -2, 2 \rangle$. The rest is $\langle 2, -1, 3 \rangle - \langle 2, -2, 2 \rangle = \langle 0, 1, 1 \rangle$, which is perpendicular to $\overrightarrow{PR}$:

$$\boxed{\overrightarrow{PQ} = \langle 2, -2, 2 \rangle + \langle 0, 1, 1 \rangle}.$$

Math 126 Final, Spring 2008 — Problem 4

(a) **F**: normals $\langle 2, -2, 1 \rangle$ and $\langle 1, -1, 1 \rangle$ are not parallel. (b) **F**: dot product $2 - 4 + 0 = -2 \neq 0$. (c) **F**: only $\vec{a} \perp (\vec{b} - \vec{c})$ follows. (d) **T**: both have direction $\langle -3, 1, 4 \rangle$ (and they are different lines). (e) **T**: $(1-t) - t - 7(4+7t) = 3$ gives $t = -\frac{30}{51}$. (f) **T**: $\langle 2, -8, 3 \rangle \cdot \langle -1, -1, -2 \rangle = 0$, and $(2, 5, -4)$ is not on the plane. (g) **T**: $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$. (h) **F**: the right side crosses two scalars, which isn't defined. (i) **T**. (j) **F**: e.g. the x - and y -axes are both perpendicular to the z -axis.

Math 126 Final, Spring 2008 — Problem 5

(a) $x = (1 + \cos \theta) \cos \theta$, $y = (1 + \cos \theta) \sin \theta$.

$$\frac{dx}{d\theta} = -\sin \theta \cos \theta - (1 + \cos \theta) \sin \theta = -\sin \theta (1 + 2 \cos \theta), \quad \frac{dy}{d\theta} = \cos \theta + \cos 2\theta.$$

$\frac{dx}{d\theta} = 0$ when $\sin \theta = 0$ or $\cos \theta = -\frac{1}{2}$: $\theta = 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$. At $\theta = \pi$ also $\frac{dy}{d\theta} = 0$ (the cusp at the origin, not a vertical tangent). Vertical tangents at $\boxed{\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}}$.

(b)

$$A = \frac{1}{2} \int_0^\pi (1 + \cos \theta)^2 d\theta = \frac{1}{2} \int_0^\pi \left(\frac{3}{2} + 2 \cos \theta + \frac{\cos 2\theta}{2} \right) d\theta = \boxed{\frac{3\pi}{4}}.$$

(c) $V = \iint_R y \, dA = \int_0^{\pi/2} \int_1^{1+\cos \theta} r \sin \theta \, r \, dr \, d\theta = \int_0^{\pi/2} \frac{(1+\cos \theta)^3 - 1}{3} \sin \theta \, d\theta$. With $u = \cos \theta$:

$$V = \frac{1}{3} \int_0^1 ((1+u)^3 - 1) du = \frac{1}{3} \left[\frac{(1+u)^4}{4} - u \right]_0^1 = \boxed{\frac{11}{12}}.$$

Math 126 Final, Spring 2008 — Problem 6

$(-\pi, 0)$ corresponds to $t = \pi$. $x' = \cos t - t \sin t$, $y' = \sin t + t \cos t$, $x'' = -2 \sin t - t \cos t$, $y'' = 2 \cos t - t \sin t$.
At $t = \pi$: $x' = -1$, $y' = -\pi$, $x'' = \pi$, $y'' = -2$.

$$\kappa = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}} = \frac{2 + \pi^2}{(1 + \pi^2)^{3/2}}, \quad R = \frac{1}{\kappa} = \boxed{\frac{(1 + \pi^2)^{3/2}}{2 + \pi^2}} \approx 3.02.$$

Math 126 Final, Spring 2008 — Problem 7

Reverse the order: $0 \leq x \leq 2$, $0 \leq y \leq 4 - x^2$ is the same as $0 \leq y \leq 4$, $0 \leq x \leq \sqrt{4 - y}$:

$$\int_0^4 \frac{e^{2y}}{4 - y} \int_0^{\sqrt{4-y}} x \, dx \, dy = \int_0^4 \frac{e^{2y}}{4 - y} \cdot \frac{4 - y}{2} dy = \frac{1}{2} \int_0^4 e^{2y} dy = \boxed{\frac{e^8 - 1}{4}}.$$

Math 126 Final, Spring 2008 — Problem 8

$$M(a) = \int_1^2 \int_{ax}^{2ax} \left(\frac{1}{x} + \frac{1}{y^2} \right) dy \, dx = \int_1^2 \left[a + \frac{1}{ax} - \frac{1}{2ax} \right] dx = \int_1^2 \left(a + \frac{1}{2ax} \right) dx = a + \frac{\ln 2}{2a}.$$

$$M'(a) = 1 - \frac{\ln 2}{2a^2} = 0 \Rightarrow \boxed{a = \sqrt{\frac{\ln 2}{2}} \approx 0.589} \text{ (a minimum, since } M'' > 0 \text{)}.$$

Math 126 Final, Spring 2008 — Problem 9

Maximize $P = xy^2z^3$ with $x + y + z = 100$. Using $\ln P = \ln x + 2 \ln y + 3 \ln z$ and Lagrange: $\frac{1}{x} = \frac{2}{y} = \frac{3}{z} = \lambda$, so $y = 2x$, $z = 3x$ and $6x = 100$:

$$x = \frac{50}{3}, \quad y = \frac{100}{3}, \quad z = 50.$$