

Math 126 Final, Winter 2008 — Problem 1

$f(-1) = -1 - 2 + 4 \cos(-\pi) = -7$. $f'(x) = 3x^2 + 2 - 4\pi \sin(\pi x)$, so $f'(-1) = 5$. $f''(x) = 6x - 4\pi^2 \cos(\pi x)$, so $f''(-1) = -6 + 4\pi^2$.

$$T_2(x) = -7 + 5(x + 1) + (2\pi^2 - 3)(x + 1)^2.$$

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$g^{(n+1)}(x) = 2^{n+1}e^{2x} \leq 2^{n+1}$ on $[-0.5, 0]$, so take $M = 2^{n+1}$. Taylor's inequality with $|x| \leq 0.5$:

$$|g - T_n| \leq \frac{2^{n+1}(0.5)^{n+1}}{(n+1)!} = \frac{1}{(n+1)!}.$$

We need $(n+1)! > 100$: $4! = 24$ is too small and $5! = 120$ works, so $\boxed{n = 4}$.

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Direction: $\langle 1, -1, 0 \rangle \times \langle 1, 2, 3 \rangle = \langle -3, -3, 3 \rangle \parallel \langle 1, 1, -1 \rangle$. A point: $y = 0$ gives $x = 3, z = -1$.

$$\boxed{x = 3 + t, \quad y = t, \quad z = -1 - t}.$$

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They collide when the positions agree: $2 \cos t = 2 \cos t$ always, and $2 \sin t = 3 - 4 \sin t \Rightarrow \sin t = \frac{1}{2}$, first at $t = \frac{\pi}{6}$.

Velocities: $\vec{v}_1 = \langle 2 \cos t, -2 \sin t \rangle = \langle \sqrt{3}, -1 \rangle$ and $\vec{v}_2 = \langle -4 \cos t, -2 \sin t \rangle = \langle -2\sqrt{3}, -1 \rangle$.

$$\cos \theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|} = \frac{-6 + 1}{2\sqrt{13}} = \boxed{-\frac{5}{2\sqrt{13}}} \approx -0.693.$$

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$D = vt$ with $v = 100$ km/h and $t = 50$ s = $\frac{50}{3600}$ h, so $D \approx 1.389$ km. Then $dD = t\,dv + v\,dt$:

$$|dD| \leq \frac{50}{3600}(2) + 100 \cdot \frac{1}{3600} = \frac{200}{3600} \approx \boxed{0.056 \text{ km} \approx 56 \text{ m}}.$$

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The triangle is $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$:

$$\begin{aligned} m &= \int_0^1 \int_0^{1-x} 3e^{-2x-3y} dy \, dx = \int_0^1 e^{-2x} (1 - e^{-3+3x}) dx = \int_0^1 (e^{-2x} - e^{-3}e^x) dx \\ &= \frac{1 - e^{-2}}{2} - e^{-3}(e - 1) = \boxed{\frac{1}{2} - \frac{3}{2}e^{-2} + e^{-3}} \approx 0.347. \end{aligned}$$

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Inner: $\int_0^{\ln y} e^{x+y} dx = e^y(y-1)$. Then, by parts,

$$I = \int_1^{\ln 8} (y-1)e^y dy = \left[(y-2)e^y\right]_1^{\ln 8} = 8(\ln 8 - 2) + e = \boxed{8\ln 8 - 16 + e} \approx 3.354.$$

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Normal at (a, b, c) : $\nabla(x^2 + y^2 + z^2) = \langle 2a, 2b, 2c \rangle$. Plane: $a(x - a) + b(y - b) + c(z - c) = 0$, and $a^2 + b^2 + c^2 = 1$, so

$ax + by + cz = 1$

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$\vec{v} = \langle \cos t, -\sin t, 1 \rangle$, $\vec{a} = \langle -\sin t, -\cos t, 0 \rangle$. Then $\vec{v} \cdot \vec{a} = 0$ for all t , so $a_T = 0$ and

$$a_N = |\vec{a}| = \boxed{1}.$$

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Lagrange with $g = x^4y^2z = 1$ ($x, y, z > 0$): $2x = \lambda \cdot 4x^3y^2z$, $2y = \lambda \cdot 2x^4yz$, $2z = \lambda x^4y^2$. Multiply by x, y, z and use $g = 1$: $x^2 = 2\lambda$, $y^2 = \lambda$, $z^2 = \frac{\lambda}{2}$. Then $x^4y^2z = 4\lambda^2 \cdot \lambda \cdot \sqrt{\lambda/2} = 1 \Rightarrow \lambda = 2^{-3/7}$:

$x \approx 1.219, \quad y \approx 0.862, \quad z \approx 0.610$

$(M_{\min} = \frac{7}{2}\lambda \approx 2.600).$