

Math 207 Midterm 1, Spring 2022 (Loveless) — Problem 1

(a) Separate: $e^{y-1}dy = \sin(4x) dx \Rightarrow e^{y-1} = -\frac{\cos 4x}{4} + C$. $y(0) = 1$: $1 = -\frac{1}{4} + C$, so $C = \frac{5}{4}$:

$$y = 1 + \ln\left(\frac{5 - \cos 4x}{4}\right).$$

(b) $y' + \left(\frac{1}{t} + 2\right)y = \frac{12}{t}$. Integrating factor $e^{\ln t + 2t} = te^{2t}$:

$$(te^{2t}y)' = 12e^{2t} \Rightarrow te^{2t}y = 6e^{2t} + C \Rightarrow y = \frac{6}{t} + \frac{Ce^{-2t}}{t}.$$

Math 207 Midterm 1, Spring 2022 (Loveless) — Problem 2

(a) $y' = e^x + r + 2sx$, $y'' = e^x + 2s$, so $y'' - y' = (2s - r) - 2sx = 8x + 10$. Then $-2s = 8$ and $2s - r = 10$:
 $r = -18$, $s = -4$.

(b) i. Equilibria: $y = 0, 8, 10$. Sign of $f = y(10 - y)(y - 8)^2$: negative for $y < 0$, positive on $(0, 8)$ and $(8, 10)$, negative for $y > 10$.

$y = 0$: **unstable**; $y = 8$: **semistable**; $y = 10$: **stable**.

ii. Starting at 9 (between 8 and 10, $f > 0$), the solution rises to the stable equilibrium: $\lim_{t \rightarrow \infty} y(t) = \boxed{10}$.

Math 207 Midterm 1, Spring 2022 (Loveless) — Problem 3

$T' + \frac{1}{4}T = 15 + 60e^{-t}$. Integrating factor $e^{t/4}$:

$$(Te^{t/4})' = 15e^{t/4} + 60e^{-3t/4} \Rightarrow Te^{t/4} = 60e^{t/4} - 80e^{-3t/4} + C \Rightarrow T = 60 - 80e^{-t} + Ce^{-t/4}.$$

$T(0) = 400$: $C = 420$:

$T(t) = 60 - 80e^{-t} + 420e^{-t/4}.$

Math 207 Midterm 1, Spring 2022 (Loveless) — Problem 4

(a) $f(t, y) = 2y^2 - 6t$, $h = \frac{1}{2}$. Step 1: $f(0, 1) = 2$, so $y(0.5) \approx 1 + \frac{1}{2}(2) = 2$. Step 2: $f(0.5, 2) = 8 - 3 = 5$, so $y(1) \approx 2 + \frac{1}{2}(5) = \boxed{4.5}$.

(b) i. $\boxed{\frac{dB}{dt} = 0.03B - K, \quad B(0) = 100,000}$.

ii. The equilibrium is $B = \frac{K}{0.03}$, and it is unstable: the balance grows if it starts above it. So the balance increases when $100,000 > \frac{K}{0.03}$, i.e. $\boxed{K < 3000 \text{ dollars per year}}$.

Math 207 Midterm 1, Spring 2022 (Loveless) — Problem 5

(a) Volume $V(t) = 100 + 30t$. Salt in: $5 \cdot 50 = 250$ g/min; out: $20 \cdot \frac{y}{100+30t}$:

$$\boxed{\frac{dy}{dt} = 250 - \frac{20y}{100 + 30t}, \quad y(0) = 20}.$$

(b) $V = x^3$ and surface area $6x^2 = 6V^{2/3}$:

$$\boxed{\frac{dV}{dt} = -1.2 V^{2/3}, \quad V(0) = 1000}.$$