

# Math 208D Final Exam — Answers

1. (This is similar to #1 and #2 in the Chapter 5 conceptual problems.) If we switch the first and second rows and then subtract or add suitable multiples of the first row from the other rows so that the first column under the pivotal 1 has only zeros, we find that  $A$  has determinant

$$-\det \begin{pmatrix} \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix} \end{pmatrix} = -5 \cdot 3 \cdot 2 = -30.$$

Then  $\det \left( \frac{1}{10} A^3 \right) = 10^{-4} (-30)^3 = -2.7$ .

2. (This is similar to #1 in the Chapter 4 conceptual problems and #4 in the Chapter 6 conceptual problems.) The coefficient vector  $\bar{a}$  for the equation  $a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$  of the plane  $P$  is perpendicular to the plane  $P$ , and so the reflection takes that vector to its negative. This means that  $[a_1 \ a_2 \ a_3]^T$  is a solution to  $(A + I_3)\bar{a} = 0$ . After a few row operations we find the reduced-echelon form of  $A + I_3$ :

$$\begin{aligned} A + I_3 &= \frac{1}{9} \begin{bmatrix} 10 & 8 & -4 \\ 8 & 10 & 4 \\ -4 & 4 & 16 \end{bmatrix} \sim \begin{bmatrix} 10 & 8 & -4 \\ 8 & 10 & 4 \\ 1 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -4 \\ 0 & 18 & 36 \\ 0 & 18 & 36 \end{bmatrix} \sim \\ &\sim \begin{bmatrix} 1 & -1 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Remember that our unknowns here are  $a_1, a_2, a_3$ . The free variable is  $a_3 = s$ , and then the two nonzero rows give us  $a_1 = 2s$  and  $a_2 = -2s$ . We can choose  $s = 1$  for simplicity (since we need only one normal vector to the plane in order to write the plane's equation). Thus, we take  $[a_1 \ a_2 \ a_3]^T = [2 \ -2 \ 1]^T$ , and the equation of the plane of reflection is  $2x_1 - 2x_2 + x_3 = 0$ .

3. (This is similar to #3 in the Chapter 6 conceptual problems.) (a) Since  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  is a vector on the line with slope 2 (and so is an eigenvector for eigenvalue 0), and  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$  is a perpendicular vector (and hence a fixed vector), we can write

$$A = PDP^{-1} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}.$$

$$(b) (A + I_2)^4 = P(D + I_2)^4 P^{-1} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}.$$

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4. (This is similar to #7 in the Chapter 6 conceptual problems.)

(a)  $\det(A - \lambda I_5) = (3 - \lambda)(0 - \lambda)^3(-1 - \lambda) = -\lambda^3(\lambda + 1)(\lambda - 3)$ .

(b) nullity=0; set  $\lambda = 1$  in part (a): determinant=-2(-2) = 4. (c) nullity=1; determinant=0; (d) no; (e) yes; (f) yes (NOTE: in (f) a correct answer means the same answer as in part (e)).

5. (This is similar to #8 in the Chapter 5 conceptual problems and #9(b) in the Chapter 6 conceptual problems.)

(a)  $A = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ . (b)  $A^2 = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ .

(c)  $(\lambda + 1)^4$ . (d)  $\lambda = -1$ . (e)  $A^2 - \lambda I_4 = A^2 + I_4 = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(in reduced echelon form). The solution space consists of  $\bar{x} = s_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

for any values of the parameters  $s_1$  and  $s_2$ .

(f)  $f(t) = x_1 \sin(t) + x_2 \cos(t)$  for any values of  $x_1, x_2$ . The calculus fact is that  $\frac{d^2}{dt^2} \sin(t) = -\sin(t)$  and the same for  $\cos(t)$ .