

Math 208 Final, Spring 2022 — Problem 1

(i) Cofactor expansion along the first row:

$$\begin{aligned}\det \begin{bmatrix} x & 1 & 0 \\ y & 2 & 1 \\ z & 0 & 3 \end{bmatrix} &= x \det \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} y & 1 \\ z & 3 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} y & 2 \\ z & 0 \end{bmatrix} \\ &= x(6 - 0) - (3y - z) + 0 = \boxed{6x - 3y + z}.\end{aligned}$$

(ii) The columns of that matrix are (x, y, z) , $(1, 2, 0)$ and $(0, 1, 3)$. The point (x, y, z) lies in P exactly when it is a linear combination of $(1, 2, 0)$ and $(0, 1, 3)$, i.e. when the three columns are linearly dependent, i.e. when the determinant is 0. So $(x, y, z) \in P \iff 6x - 3y + z = 0$:

$$\boxed{a = 6, \quad b = -3, \quad c = 1}.$$

Check: $(1, 2, 0)$: $6 - 6 + 0 = 0$; $(0, 1, 3)$: $0 - 3 + 3 = 0$. Both spanning vectors satisfy the equation.

Math 208 Final, Spring 2022 — Problem 2

(a) $A - \lambda I$ is upper triangular, so its determinant is the product of the diagonal entries:

$$\det(A - \lambda I) = (2 - \lambda)(2 - \lambda)(3 - \lambda).$$

The eigenvalues are $\lambda = 2$ (multiplicity 2) and $\lambda = 3$.

(b) $\lambda = 2$: row reduce $A - 2I$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_2 = 0$, $x_3 = 0$ and x_1 is free. Eigenspace basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

$\lambda = 3$: row reduce $A - 3I$.

$$\begin{aligned} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} &\xrightarrow{-R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ &\xrightarrow{-R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ &\xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So $x_1 = x_3$, $x_2 = x_3$ with x_3 free. Eigenspace basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Check: $A(1, 0, 0) = (2, 0, 0) = 2(1, 0, 0)$ and $A(1, 1, 1) = (3, 3, 3) = 3(1, 1, 1)$.

(c) **No.** The eigenspaces have dimensions 1 and 1, so there are only 2 linearly independent eigenvectors, but a 3×3 matrix needs 3 to be diagonalizable.

Math 208 Final, Spring 2022 — Problem 3

Track how each row operation changes the determinant:

- $R_1 \leftrightarrow R_2$ (swap): $\det A_1 = -\det A$.
- $R_3 - R_1 \rightarrow R_3$ and $R_3 - R_2 \rightarrow R_3$ (row replacements): no change, so $\det A_3 = \det A_1 = -\det A$.
- $\frac{1}{2}R_3 \rightarrow R_3$ (scaling a row by $\frac{1}{2}$): $\det B = \frac{1}{2} \det A_3 = -\frac{1}{2} \det A$.

B is upper triangular, so $\det B = 1 \cdot 1 \cdot x = x$. Therefore

$$-\frac{1}{2} \det A = x \implies \boxed{\det A = -2x}.$$

Math 208 Final, Spring 2022 — Problem 4

The characteristic polynomial $\lambda^3(\lambda - 1)^2(\lambda + 1)$ has degree 6, so A is 6×6 . Its eigenvalues are 0 (algebraic multiplicity 3), 1 (multiplicity 2) and -1 (multiplicity 1).

(a) $\boxed{-1, 0, 1}$.

(b) $\text{nullity}(A) = \dim \text{Nul}(A - 0I)$, the geometric multiplicity of $\lambda = 0$. It is at least 1 (0 is an eigenvalue) and at most the algebraic multiplicity 3: $\boxed{1, 2, 3}$.

(c) Find the null space of $A^2 - I = (A - I)(A + I)$. If $A^2v = v$, write $v = \frac{1}{2}(v + Av) + \frac{1}{2}(v - Av)$. Then

$$(A - I)(v + Av) = A^2v - v = 0, \quad (A + I)(v - Av) = v - A^2v = 0,$$

so $v + Av$ is in the eigenspace E_1 and $v - Av$ is in E_{-1} . Conversely every vector of E_1 or E_{-1} satisfies $A^2v = v$. Hence

$$\text{nullity}(A^2 - I) = \dim E_1 + \dim E_{-1}.$$

Here $\dim E_{-1} = 1$ and $\dim E_1 \in \{1, 2\}$, so the nullity is 2 or 3 and, by the rank theorem, $\text{rank}(A^2 - I) = 6 - \text{nullity} \in \boxed{\{3, 4\}}$.

(Equivalently: the eigenvalues of A^2 are 0, 0, 0, 1, 1, 1, and $A^2 - I$ has a zero eigenvalue exactly for the $\lambda = \pm 1$ eigenvectors of A .)

Math 208 Final, Spring 2022 — Problem 5

(a) Read the coefficients of x, y, z, w in each output:

$$A = \begin{bmatrix} 1 & 0 & 4 & -1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

(b) Solve $A\vec{x} = \vec{0}$. The matrix is already in reduced echelon form (pivots in columns 1 and 2), so z and w are free:

$$x = -4z + w, \quad y = -z - 2w,$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = z \begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \quad \ker T = \operatorname{span} \left\{ \begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Check: $T(-4, -1, 1, 0) = (-4+4-0, -1+1+0) = (0, 0)$ and $T(1, -2, 0, 1) = (1+0-1, -2+0+2) = (0, 0)$.

(c) **No.** The kernel contains nonzero vectors (it is 2-dimensional), so two different inputs (e.g. $\vec{0}$ and $(-4, -1, 1, 0)$) have the same image. (Any linear map $\mathbb{R}^4 \rightarrow \mathbb{R}^2$ has a kernel of dimension at least $4-2 = 2$.)

Math 208 Final, Spring 2022 — Problem 6

(a) Multiply:

$$A \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \cdot 2 + 2x + 2y \\ 2x + 2 + 0 \\ 2y + 0 + 2 \end{bmatrix} = \begin{bmatrix} 2x + 2y \\ 2x + 2 \\ 2y + 2 \end{bmatrix}.$$

For $(2, 2, 2)$ to be an eigenvector we need this to equal $\lambda(2, 2, 2)$ for some λ , i.e. $x + y = \lambda$, $x + 1 = \lambda$, $y + 1 = \lambda$. The last two give $x = y$; then $2x = x + 1$ gives $x = 1$:

$$\boxed{x = 1, \quad y = 1} \quad (\lambda = 2).$$

Check: with $x = y = 1$, $A(2, 2, 2) = (4, 4, 4) = 2(2, 2, 2)$.

(b) Use $\det(A^2) = \det(A) \det(A) = (\det A)^2$:

$$\det(A^2) = (-x^2 - y^2)^2 = (x^2 + y^2)^2.$$

For real numbers, $(x^2 + y^2)^2 = 0 \iff x^2 + y^2 = 0 \iff \boxed{x = 0 \text{ and } y = 0}$.

Math 208 Final, Spring 2022 — Problem 7

(a) **Not possible.** By linearity, $T((1, 1)) - T((1, 0)) = T((0, 1))$, so $T(0, 1) = (2, 2) - (3, 1) = (-1, 1)$. Then

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = 2T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

(b) **Not possible.** A 2×2 matrix has a degree-2 characteristic polynomial, so it has at most 2 eigenvalues.

(c) For example $B = \begin{bmatrix} 8 & 0 & 0 \\ 8 & 0 & 0 \\ 8 & 0 & 0 \end{bmatrix}$.

Check: $B \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\det B = 0$ because B has a column of zeros.

Math 208 Final, Spring 2022 — Problem 8

(a) The mistake is the first step, $\det(A - \lambda I) = \det A - \det(\lambda I)$. The determinant is *not* additive: $\det(X - Y) \neq \det X - \det Y$ in general. (Everything after that step depends on it.)

(b) For example $A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

Check: $\det A = 2 \cdot \frac{1}{2} = 1$, and since A is diagonal its eigenvalues are 2 and $\frac{1}{2}$, neither of which is ± 1 .

Math 208 Final, Spring 2022 — Problem 9

(a) Multiply each given eigenvector by A :

$$A \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6+3 \\ 3+0 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2-3 \\ 1-0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Put the eigenvectors in the columns of P and the matching eigenvalues on the diagonal of D :

$$P = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Check: $AP = \begin{bmatrix} 9 & -1 \\ 3 & 1 \end{bmatrix} = PD.$

(b) **Not possible.** If C is diagonalizable with real eigenvalues, $C = QEQ^{-1}$ with E real diagonal. Then $C^2 = QE^2Q^{-1}$, so every eigenvalue of C^2 is the square of a real number and hence ≥ 0 . But A has the eigenvalue $-1 < 0$, so $C^2 = A$ is impossible.

(c) Solve $c_1(3, 1) + c_2(1, -1) = (1, 0)$ by row reducing the augmented matrix:

$$\begin{aligned} \left[\begin{array}{cc|c} 3 & 1 & 1 \\ 1 & -1 & 0 \end{array} \right] &\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & 1 & 1 \end{array} \right] \\ &\xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 4 & 1 \end{array} \right] \\ &\xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & \frac{1}{4} \end{array} \right] \\ &\xrightarrow{R_1 + R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} \end{array} \right] \end{aligned}$$

So $c_1 = c_2 = \frac{1}{4}$: $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix}.$

Check: $\frac{1}{4}(3, 1) + \frac{1}{4}(1, -1) = (1, 0).$

(d) **No.** A subspace must contain the zero vector, but by definition $\vec{0}$ is not an eigenvector, so $\vec{0} \notin S$.

Math 208 Final, Spring 2022 — Problem 10

(a) $3B = A + I$, so $A = 3B - I$. If $B\vec{v} = \mu\vec{v}$ then $A\vec{v} = 3B\vec{v} - \vec{v} = (3\mu - 1)\vec{v}$. The eigenvalues of B are 0 and 1, so the eigenvalues of A are

$$3(0) - 1 = -1 \quad \text{and} \quad 3(1) - 1 = 2: \quad \boxed{\lambda = -1, 2}.$$

Check: the product of the eigenvalues (with multiplicity) must be $\det A = 2$, which works as $(-1)(-1)(2) = 2$; and their sum must be $\text{tr } A = 0$: $-1 - 1 + 2 = 0$.

(b) Write $A_n = J - I$ where J is the $n \times n$ matrix of all ones. $J(1, \dots, 1) = n(1, \dots, 1)$, and $J\vec{v} = \vec{0}$ for every $\vec{v}$ whose entries sum to 0 (an $(n - 1)$ -dimensional space). So J has eigenvalues n (once) and 0 ($n - 1$ times), and $A_n = J - I$ has eigenvalues $n - 1$ (once) and -1 ($n - 1$ times). The determinant is their product:

$$\boxed{\det A_n = (n - 1)(-1)^{n-1}}.$$

Check: $n = 2$: $\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 = (1)(-1)^1$; $n = 3$: $\det A = 2 = (2)(-1)^2$.