

Math 208 Final, Spring 2025 — Problem 1

One elementary row operation at a time:

$$\begin{aligned} \begin{pmatrix} 2 & 0 & 8 & 0 & 4 \\ -1 & 1 & -1 & 1 & x \\ 0 & 1 & 3 & 2 & x+2 \end{pmatrix} &\xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & 4 & 0 & 2 \\ -1 & 1 & -1 & 1 & x \\ 0 & 1 & 3 & 2 & x+2 \end{pmatrix} \\ &\xrightarrow{R_2+R_1} \begin{pmatrix} 1 & 0 & 4 & 0 & 2 \\ 0 & 1 & 3 & 1 & x+2 \\ 0 & 1 & 3 & 2 & x+2 \end{pmatrix} \\ &\xrightarrow{R_3-R_2} \begin{pmatrix} 1 & 0 & 4 & 0 & 2 \\ 0 & 1 & 3 & 1 & x+2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\ &\xrightarrow{R_2-R_3} \begin{pmatrix} 1 & 0 & 4 & 0 & 2 \\ 0 & 1 & 3 & 0 & x+2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

Every pivot is 1 and is the only nonzero entry in its column, so this is the RREF:

$$\boxed{\begin{pmatrix} 1 & 0 & 4 & 0 & 2 \\ 0 & 1 & 3 & 0 & x+2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}}.$$

Math 208 Final, Spring 2025 — Problem 2

Yes. First, $\vec{v}$ and $\vec{w}$ are linearly independent (neither is a multiple of the other: $\vec{w}$ has a 0 where $\vec{v}$ has a 3). Try adding $\vec{e}_3$ and $\vec{e}_4$ and row reduce $[\vec{v} \ \vec{w} \ \vec{e}_3 \ \vec{e}_4]$:

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} &\xrightarrow{R_2-3R_1, \ R_3-2R_1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{R_1-R_2, \ R_4-2R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

There is a pivot in every column, so $\{\vec{v}, \vec{w}, \vec{e}_3, \vec{e}_4\}$ is linearly independent. Four linearly independent vectors in $\mathbb{R}^4$ form a basis:

$$\left\{ \vec{v}, \vec{w}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Check: the same matrix has determinant (expand along column 4, then column 3) $\det \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} = -3 \neq 0$.

Math 208 Final, Spring 2025 — Problem 3

Cofactor expansion along the first row:

$$\det B = 2 \det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - 1 \det \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + 1 \det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = 2(3) - 1(1) + 1(-1) = 4 \neq 0,$$

so B is invertible. Row reduce $[B \mid I]$:

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array} \right] &\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array} \right] \\ &\xrightarrow{R_2 - 2R_1, R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -3 & -1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array} \right] \\ &\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & -3 & -1 & 1 & -2 & 0 \end{array} \right] \\ &\xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & -3 & -1 & 1 & -2 & 0 \end{array} \right] \\ &\xrightarrow{R_1 - 2R_2, R_3 + 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & -4 & 1 & 1 & -3 \end{array} \right] \\ &\xrightarrow{-\frac{1}{4}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array} \right] \\ &\xrightarrow{R_1 - 3R_3, R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array} \right] \end{aligned}$$

The left block is now I , so the right block is B^{-1} :

$$B^{-1} = \frac{1}{4} \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

Check: $BB^{-1} = \frac{1}{4} \begin{pmatrix} 6 - 1 - 1 & -2 + 3 - 1 & -2 - 1 + 3 \\ 3 - 2 - 1 & -1 + 6 - 1 & -1 - 2 + 3 \\ 3 - 1 - 2 & -1 + 3 - 2 & -1 - 1 + 6 \end{pmatrix} = I.$

Math 208 Final, Spring 2025 — Problem 4

Write C for the matrix. Expand $\det(C - \lambda I)$ along columns/rows with a single nonzero entry:

$$\begin{aligned}\det(C - \lambda I) &= (-1 - \lambda) \det \begin{pmatrix} -41 - \lambda & 3 & 12 & 2 \\ 0 & 3 - \lambda & 1 & 6 \\ 0 & 0 & 2 - \lambda & 0 \\ 0 & 2 & 2 & -1 - \lambda \end{pmatrix} && \text{(column 1)} \\ &= (-1 - \lambda)(-41 - \lambda) \det \begin{pmatrix} 3 - \lambda & 1 & 6 \\ 0 & 2 - \lambda & 0 \\ 2 & 2 & -1 - \lambda \end{pmatrix} && \text{(column 1)} \\ &= (-1 - \lambda)(-41 - \lambda)(2 - \lambda) \det \begin{pmatrix} 3 - \lambda & 6 \\ 2 & -1 - \lambda \end{pmatrix} && \text{(row 2)} \\ &= (-1 - \lambda)(-41 - \lambda)(2 - \lambda)[(3 - \lambda)(-1 - \lambda) - 12] \\ &= (-1 - \lambda)(-41 - \lambda)(2 - \lambda)(\lambda^2 - 2\lambda - 15) = (-1 - \lambda)(-41 - \lambda)(2 - \lambda)(\lambda - 5)(\lambda + 3).\end{aligned}$$

Eigenvalues: $\boxed{-1, -41, 2, 5, -3}$.

Check: the eigenvalues must add up to the trace: $-1 - 41 + 2 + 5 - 3 = -38 = (-1) + (-41) + 3 + 2 + (-1)$.

Math 208 Final, Spring 2025 — Problem 5

A vector $\vec{x}$ is orthogonal to the column span of D exactly when it is orthogonal to both columns, i.e. $D^T \vec{x} = \vec{0}$. Row reduce D^T :

$$\begin{aligned} \begin{bmatrix} 1 & 3 & 0 & 0 \\ 1 & 0 & 3 & 6 \end{bmatrix} &\xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & -3 & 3 & 6 \end{bmatrix} \\ &\xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & -2 \end{bmatrix} \\ &\xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 3 & 6 \\ 0 & 1 & -1 & -2 \end{bmatrix} \end{aligned}$$

x_3, x_4 are free: $x_1 = -3x_3 - 6x_4$, $x_2 = x_3 + 2x_4$. So

$$S = \text{span} \left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Check: with the columns $(1, 3, 0, 0)$ and $(1, 0, 3, 6)$: $(-3, 1, 1, 0)$ gives $-3 + 3 = 0$ and $-3 + 3 = 0$; $(-6, 2, 0, 1)$ gives $-6 + 6 = 0$ and $-6 + 0 + 6 = 0$.

Math 208 Final, Spring 2025 — Problem 7

Find all (w, x, y, z) with $T(w, x, y, z) = \vec{0}$, solving one equation at a time:

- $2w = 0 \Rightarrow w = 0$.
- $5w - x = 0 \Rightarrow x = 0$.
- $7w + 2x + y = 0 \Rightarrow y = 0$.
- $13w + 41x + 11y + 4z = 0 \Rightarrow 4z = 0 \Rightarrow z = 0$.

Only the zero vector is in the kernel, so $\boxed{\dim \ker T = 0}$.

Check: the matrix of T (variables in the order w, x, y, z) is $\begin{bmatrix} 5 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 7 & 2 & 1 & 0 \\ 13 & 41 & 11 & 4 \end{bmatrix}$, whose determinant is $8 \neq 0$

(expand along the last column, then the third: $4 \cdot 1 \cdot \det \begin{bmatrix} 5 & -1 \\ 2 & 0 \end{bmatrix} = 4 \cdot 2 = 8$). An invertible matrix has rank 4 and nullity 0.

Math 208 Final, Spring 2025 — Problem 8

Eigenvalues. G is similar to B , so it has the same characteristic polynomial:

$$\det(G - \lambda I) = \det(A(B - \lambda I)A^{-1}) = \det A \det(B - \lambda I) \frac{1}{\det A} = (2 - \lambda)^2(-1 - \lambda).$$

So the eigenvalues are $\lambda = 2$ (twice) and $\lambda = -1$.

Eigenvectors. If $B\vec{u} = \lambda\vec{u}$, then $G(A\vec{u}) = ABA^{-1}A\vec{u} = AB\vec{u} = \lambda(A\vec{u})$. The eigenvectors of the diagonal matrix B are $\vec{e}_1, \vec{e}_2$ (for 2) and $\vec{e}_3$ (for -1), so the eigenvectors of G are $A\vec{e}_1, A\vec{e}_2, A\vec{e}_3$, the columns of A . Since $\det A = 1 \neq 0$ these are independent:

$$E_2 = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}, \quad E_{-1} = \text{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Check: computing $G = ABA^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -21 & 9 & -1 \end{bmatrix}$ directly: $G(1, 2, -1) = (2, 4, -2)$, $G(0, 1, 3) = (0, 2, 6)$,
 $G(0, 0, 1) = (0, 0, -1)$.

Math 208 Final, Spring 2025 — Problem 9

(The question says “ A ”; it means F .) The eigenvalues are 0, 1, 2. For each, row reduce $F - \lambda I$ and solve for the null space.

$\lambda = 0$:

$$\begin{aligned} \begin{bmatrix} 2 & 5 & 1 \\ 0 & -3 & -1 \\ 2 & 14 & 4 \end{bmatrix} &\xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & \frac{5}{2} & \frac{1}{2} \\ 0 & -3 & -1 \\ 2 & 14 & 4 \end{bmatrix} \\ &\xrightarrow{R_3-2R_1} \begin{bmatrix} 1 & \frac{5}{2} & \frac{1}{2} \\ 0 & -3 & -1 \\ 0 & 9 & 3 \end{bmatrix} \\ &\xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & \frac{5}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ 0 & 9 & 3 \end{bmatrix} \\ &\xrightarrow{R_1-\frac{5}{2}R_2, R_3-9R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Basis: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right\}.$

$\lambda = 1$:

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & -4 & -1 \\ 2 & 14 & 3 \end{bmatrix} \xrightarrow{R_3-2R_1} \begin{bmatrix} 1 & 5 & 1 \\ 0 & -4 & -1 \\ 0 & 4 & 1 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{4}R_2} \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & \frac{1}{4} \\ 0 & 4 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1-5R_2, R_3-4R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

Basis: $\boxed{\left\{ \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\}}.$

$\lambda = 2$:

$$\begin{bmatrix} 0 & 5 & 1 \\ 0 & -5 & -1 \\ 2 & 14 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 2 & 14 & 2 \\ 0 & -5 & -1 \\ 0 & 5 & 1 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 7 & 1 \\ 0 & -5 & -1 \\ 0 & 5 & 1 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{5}R_2} \begin{bmatrix} 1 & 7 & 1 \\ 0 & 1 & \frac{1}{5} \\ 0 & 5 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1-7R_2, R_3-5R_2} \begin{bmatrix} 1 & 0 & -\frac{2}{5} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

Basis: $\boxed{\left\{ \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \right\}}.$

Check: $F(1, -1, 3) = (0, 0, 0)$, $F(1, -1, 4) = (1, -1, 4)$, $F(2, -1, 5) = (4, -2, 10) = 2(2, -1, 5)$.

Math 208 Final, Spring 2025 — Problem 10

The map. Multiply:

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{rotate } 180^\circ} \underbrace{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}_{\text{shear}}.$$

So $T(x, y) = (-x + y, -y)$: a horizontal shear $(x, y) \mapsto (x - y, y)$ followed by a rotation by 180° about the origin. Reading the columns: $T(1, 0) = (-1, 0)$ and $T(0, 1) = (1, -1)$.

The smiley face. With the origin at the center: the round face becomes a slanted ellipse (a shear turns a circle into an ellipse) and the whole picture is upside down. A point on the eyes at height $y > 0$, e.g. $(\pm a, b)$, goes to $(\mp a + b, -b)$: the eyes end up *below* the center and shifted right by b . The smile, which is below the center ($y < 0$), ends up above the center, curving downward like a frown, and shifted left. Points on the x -axis are only flipped: $(x, 0) \mapsto (-x, 0)$.

Bonus. Expand along the first row:

$$\det A(x) = x(x^2 - 1) - 1(x - 1) + 1(1 - x) = x^3 - 3x + 2 = (x - 1)^2(x + 2).$$

So $A(x)$ is invertible exactly when $\boxed{x \neq 1 \text{ and } x \neq -2}$. For the inverse write $A(x) = (x - 1)I + J$, where J is the all-ones 3×3 matrix, which satisfies $J^2 = 3J$. Try $A(x)^{-1} = \alpha I + \beta J$:

$$((x - 1)I + J)(\alpha I + \beta J) = (x - 1)\alpha I + (\alpha + (x - 1)\beta + 3\beta)J = I$$

requires $\alpha = \frac{1}{x-1}$ and $\alpha + (x + 2)\beta = 0$, so $\beta = -\frac{1}{(x-1)(x+2)}$:

$$\boxed{A(x)^{-1} = \frac{1}{x-1} \left(I - \frac{1}{x+2} J \right) = \frac{1}{(x-1)(x+2)} \begin{pmatrix} x+1 & -1 & -1 \\ -1 & x+1 & -1 \\ -1 & -1 & x+1 \end{pmatrix}}.$$

Check: at $x = 0$ this gives $-\frac{1}{2} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$, and multiplying by $A(0) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ gives I .