

Math 208 Final, Winter 2019 — Problem 1

(a) Row reduce $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$:

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} &\xrightarrow{R_2-2R_1, \ R_3-2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 1 & 2 \end{bmatrix} \\ &\xrightarrow{R_1-2R_2, \ R_3+2R_2, \ R_4-R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ &\xrightarrow{R_1+R_3, \ R_2-R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

There is a pivot in every column, so $A\vec{c} = \vec{0}$ has only the trivial solution: the set is **linearly independent**.

(b) Three linearly independent vectors form a basis of their span, so $\dim \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = 3$.

Math 208 Final, Winter 2019 — Problem 2

(a)

$$A\mathbf{x} = \begin{bmatrix} 2(-5) + 3(3) + 0(1) + 1(1) \\ 1(-5) + 1(3) + 1(1) + 1(1) \end{bmatrix} = \begin{bmatrix} -10 + 9 + 0 + 1 \\ -5 + 3 + 1 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so $\mathbf{x} \in \text{Nul } A$.

(b) Row reduce A to find the dimension of the null space:

$$\begin{aligned} \begin{bmatrix} 2 & 3 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} &\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -1 \end{bmatrix} \\ &\xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & -1 \end{bmatrix} \end{aligned}$$

Two pivots and two free variables (x_3, x_4) , so $\dim \text{Nul } A = 2$, with basis $\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ (notice $\mathbf{x}$ is their sum). In a 2-dimensional space any two linearly independent vectors form a basis, so

$$\left\{ \mathbf{x}, \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis containing $\mathbf{x}$: both are in $\text{Nul } A$ and neither is a multiple of the other.

Math 208 Final, Winter 2019 — Problem 3

(a) The coordinate vector gives the weights on the basis vectors:

$$\mathbf{x} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 - 3 \\ 2 - 3 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}.$$

(b) Solve $c_1(2, 1) + c_2(1, 1) = (4, 3)$:

$$\begin{aligned} \left[\begin{array}{cc|c} 2 & 1 & 4 \\ 1 & 1 & 3 \end{array} \right] &\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 4 \end{array} \right] \\ &\xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1 & -2 \end{array} \right] \\ &\xrightarrow{-R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array} \right] \\ &\xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array} \right] \end{aligned}$$

So $\boxed{[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}.$

Check: $1(2, 1) + 2(1, 1) = (4, 3).$

Math 208 Final, Winter 2019 — Problem 4

Multiply both sides of $AX(D + BX)^{-1} = C$ on the right by $D + BX$:

$$AX = C(D + BX) = CD + CBX.$$

Collect the X terms on the left (keeping the matrices on the left of X):

$$AX - CBX = CD \implies (A - CB)X = CD \implies \boxed{X = (A - CB)^{-1}CD}.$$

Math 208 Final, Winter 2019 — Problem 5

Put the inputs and outputs side by side as columns: $A \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$. The input matrix has determinant $1 - 4 = -3$, so

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{-3} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \cdot \frac{1}{-3} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \frac{1}{-3} \begin{bmatrix} -3 & 0 \\ 6 & -9 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}}.$$

Check: $A(1, 2) = (1, -2 + 6) = (1, 4)$ and $A(2, 1) = (2, -4 + 3) = (2, -1)$.

Math 208 Final, Winter 2019 — Problem 6

Row reduce $[\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{v}_1 \ \mathbf{v}_2]$:

$$\begin{aligned} \begin{bmatrix} 2 & 3 & -4 & 13 \\ 1 & 1 & -1 & 5 \\ -1 & -3 & 5 & -11 \end{bmatrix} &\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & -1 & 5 \\ 2 & 3 & -4 & 13 \\ -1 & -3 & 5 & -11 \end{bmatrix} \\ &\xrightarrow{R_2 - 2R_1, \ R_3 + R_1} \begin{bmatrix} 1 & 1 & -1 & 5 \\ 0 & 1 & -2 & 3 \\ 0 & -2 & 4 & -6 \end{bmatrix} \\ &\xrightarrow{R_1 - R_2, \ R_3 + 2R_2} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Reading the last two columns: $\mathbf{v}_1 = \mathbf{u}_1 - 2\mathbf{u}_2$ and $\mathbf{v}_2 = 2\mathbf{u}_1 + 3\mathbf{u}_2$. So $\mathbf{v}_1, \mathbf{v}_2 \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

Both spans are 2-dimensional: $\mathbf{u}_1, \mathbf{u}_2$ are independent (pivots in columns 1 and 2), and $\mathbf{v}_1, \mathbf{v}_2$ are independent because their coefficient matrix $\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$ has determinant $3 + 4 = 7 \neq 0$. A 2-dimensional subspace inside another 2-dimensional subspace is the whole thing, so **yes, the spans are equal**.

Check: $\mathbf{u}_1 - 2\mathbf{u}_2 = (2 - 6, 1 - 2, -1 + 6) = (-4, -1, 5) = \mathbf{v}_1$; $2\mathbf{u}_1 + 3\mathbf{u}_2 = (4 + 9, 2 + 3, -2 - 9) = (13, 5, -11) = \mathbf{v}_2$.

Math 208 Final, Winter 2019 — Problem 7

Eigenvalues.

$$\det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & -1 \\ 2 & 3 - \lambda \end{bmatrix} = (1 - \lambda)(3 - \lambda) + 2 = \lambda^2 - 4\lambda + 5.$$

$$\lambda = \frac{4 \pm \sqrt{16 - 20}}{2} = \boxed{2 \pm i}.$$

Eigenvectors. For $\lambda = 2 + i$:

$$A - (2 + i)I = \begin{bmatrix} -1 - i & -1 \\ 2 & 1 - i \end{bmatrix}.$$

The rows are multiples of each other (row 2 is $(-1 + i)$ times row 1), so one equation remains: $(-1 - i)v_1 - v_2 = 0$, i.e. $v_2 = (-1 - i)v_1$. Taking $v_1 = 1$:

$$\boxed{\lambda = 2 + i : \begin{bmatrix} 1 \\ -1 - i \end{bmatrix}, \quad \lambda = 2 - i : \begin{bmatrix} 1 \\ -1 + i \end{bmatrix}}$$

(the second is the complex conjugate, since A is real; any nonzero multiples also work).

Check: $A \begin{bmatrix} 1 \\ -1 - i \end{bmatrix} = \begin{bmatrix} 1 + 1 + i \\ 2 - 3 - 3i \end{bmatrix} = \begin{bmatrix} 2 + i \\ -1 - 3i \end{bmatrix}$ and $(2 + i) \begin{bmatrix} 1 \\ -1 - i \end{bmatrix} = \begin{bmatrix} 2 + i \\ -2 - 2i - i - i^2 \end{bmatrix} = \begin{bmatrix} 2 + i \\ -1 - 3i \end{bmatrix}.$

Math 208 Final, Winter 2019 — Problem 8

(a) Expand $\det(A - \lambda I)$ along the first column:

$$\det \begin{bmatrix} 2 - \lambda & 2 & -3 \\ 0 & -\lambda & 3 \\ 0 & 2 & -1 - \lambda \end{bmatrix} = (2 - \lambda)[(-\lambda)(-1 - \lambda) - 6] = (2 - \lambda)(\lambda^2 + \lambda - 6) = (2 - \lambda)(\lambda + 3)(\lambda - 2).$$

Eigenvalues: $\lambda = 2$ (twice), $\lambda = -3$.

$\lambda = 2$: row reduce $A - 2I$.

$$\begin{bmatrix} 0 & 2 & -3 \\ 0 & -2 & 3 \\ 0 & 2 & -3 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 0 & 1 & -\frac{3}{2} \\ 0 & -2 & 3 \\ 0 & 2 & -3 \end{bmatrix} \xrightarrow{R_2+2R_1, R_3-2R_1} \begin{bmatrix} 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

One equation, $x_2 - \frac{3}{2}x_3 = 0$, with x_1, x_3 free. Basis: $\left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \right]$.

$\lambda = -3$: row reduce $A + 3I$.

$$\begin{bmatrix} 5 & 2 & -3 \\ 0 & 3 & 3 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{\frac{1}{5}R_1} \begin{bmatrix} 1 & \frac{2}{5} & -\frac{3}{5} \\ 0 & 3 & 3 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & \frac{2}{5} & -\frac{3}{5} \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{R_1 - \frac{2}{5}R_2, R_3 - 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

(b) **Yes.** The eigenspace dimensions add to $2 + 1 = 3$, so there are three independent eigenvectors:

$$A = PDP^{-1}, \quad P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Check: $AP = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 6 & 3 \\ 0 & 4 & -3 \end{bmatrix} = PD.$

Math 208 Final, Winter 2019 — Problem 9

(a) The spanning set contains $(1, 2)$ and $(-1, 2)$, which are not multiples of each other, so $\text{col}(A) = \mathbb{R}^2$: we need a 2×4 matrix of rank 2 (then nullity $= 4 - 2 = 2$). For example

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (\text{col } A = \mathbb{R}^2, \text{ nullity } 4 - 2 = 2).$$

(b) For example $A = \text{diag}(1, 1, 1, 0)$: three pivots so rank 3, and $\det A = 1 \cdot 1 \cdot 1 \cdot 0 = 0$.

(c) **Not possible.** By the rank theorem, $\dim \ker T = 4 - \text{rank} \geq 4 - 3 = 1$, so the kernel always contains a nonzero vector and T cannot be one-to-one.

Math 208 Final, Winter 2019 — Problem 10

(a) **True.** $\det(A^2) = (\det A)^2 = 0^2 = 0$.

(b) **True.** $\det A$ is the product of the eigenvalues: $4 \cdot (-1) \cdot (-1) = 4 \neq 0$. (Equivalently, 0 is not an eigenvalue.)

(c) **False.** For example $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$: $\text{row}(A) = \text{span}\{(1, 0)\}$ but $\text{col}(A) = \text{span}\{(1, 1)\}$.

(d) **False.** The characteristic polynomial of a real matrix has real coefficients, so non-real roots come in conjugate pairs. $2 + i$ would force $2 - i$ to be an eigenvalue too, giving four eigenvalues for a 3×3 matrix.

(e) **False.** With more variables than equations a *consistent* system has infinitely many solutions, but the system might be inconsistent (e.g. two equations $x_1 = 0$ and $x_1 = 1$ among the four).