

Math 208 Final, Winter 2024 — Problem 1

(a) As columns: $A \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}$. The input matrix has determinant $5 - 4 = 1$, so its inverse is $\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$ and

$$A = \begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 25 - 14 & -10 + 7 \\ 15 - 8 & -6 + 4 \end{bmatrix} = \boxed{\begin{bmatrix} 11 & -3 \\ 7 & -2 \end{bmatrix}}.$$

Check: $A(1, 2) = (11 - 6, 7 - 4) = (5, 3)$ and $A(2, 5) = (22 - 15, 14 - 10) = (7, 4)$.

(b)

- “Every $S : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ is onto”: **False** (the zero map has range $\{\vec{0}\}$).
- “No $S : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ is one-to-one”: **True** ($\dim \ker S \geq 4 - 3 = 1$).
- “If the rows of an $n \times m$ matrix span $\mathbb{R}^m$, then $T(\mathbf{x}) = A\mathbf{x}$ is onto”: **False**. The rows spanning $\mathbb{R}^m$ means rank m , but onto $\mathbb{R}^n$ needs rank n . Example: $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (its rows span $\mathbb{R}^1$, but $T(x) = (x, 0)$ misses $(0, 1)$).

Math 208 Final, Winter 2024 — Problem 2

Let $M = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$. Expand along the third row:

$$\det M = \det \begin{bmatrix} a_1 & 3 & -3 \\ 1 & 1 & -1 \\ 0 & 0 & a_2 - 1 \end{bmatrix} = (a_2 - 1) \det \begin{bmatrix} a_1 & 3 \\ 1 & 1 \end{bmatrix} = (a_2 - 1)(a_1 - 3).$$

If $\det M \neq 0$ the columns span $\mathbb{R}^3$, so $\mathbf{b}$ is in the span. Remaining cases:

- $a_1 = 3$: then $\mathbf{v}_1 = \mathbf{v}_2 = (3, 1, 0)$, and $\mathbf{v}_3 + \mathbf{v}_2 = (0, 0, a_2 - 1)$. Every vector in the span has the form $s(3, 1, 0) + t(0, 0, 1)$ (or just $s(3, 1, 0)$ if $a_2 = 1$), so its first two entries are $(3s, s)$. $\mathbf{b}$ has first two entries $(0, 2)$, which is not of that form: $\mathbf{b}$ is **not** in the span.
- $a_1 \neq 3, a_2 = 1$: $\mathbf{v}_1 = (a_1, 1, 0)$ and $\mathbf{v}_2 = (3, 1, 0)$ are independent and span the whole xy -plane, which contains $\mathbf{b} = (0, 2, 0)$.

$$\boxed{\mathbf{b} \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \iff a_1 = 3 \text{ } (a_2 \text{ arbitrary})}.$$

Math 208 Final, Winter 2024 — Problem 3

(1) Row reduce A :

$$\begin{aligned} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 3 & 2 & -3 & -2 \\ 0 & 1 & 0 & -1 \end{bmatrix} &\xrightarrow{R_2-3R_1} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \\ &\xrightarrow{-R_2} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \\ &\xrightarrow{R_1-R_2, R_3-R_2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So $x_1 = x_3$, $x_2 = x_4$, with x_3, x_4 free:

$$\ker T = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Check: $A(1, 0, 1, 0) = (1 - 1, 3 - 3, 0) = \vec{0}$ and $A(0, 1, 0, 1) = (1 - 1, 2 - 2, 1 - 1) = \vec{0}$.

(2) Use the kernel basis as the columns of $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$:

$$S(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}.$$

The range of S is the span of its columns, which is $\ker T$; and the columns are linearly independent, so $S\mathbf{x} = \vec{0}$ only for $\mathbf{x} = \vec{0}$, i.e. S is one-to-one.

Math 208 Final, Winter 2024 — Problem 4

(1) For example $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$: all entries are nonzero, and $\det A = 0$ because two rows are equal (subtracting row 1 from row 2 gives a zero row).

(2) Take the three given vectors as the rows of A ; then $\text{row}(A)$ is exactly their span. Row reduce to find the rank:

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 3 & 2 \end{bmatrix} &\xrightarrow{R_2 - R_1, R_3 - R_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \\ &\xrightarrow{R_1 - R_2, R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Rank 2, so nullity $= 4 - 2 = 2$ as required:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 3 & 2 \end{bmatrix}.$$

(3) **Impossible.** For $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, one-to-one means $\dim \ker T = 0$, so by the rank theorem $\text{rank } T = 3 - 0 = 3$, i.e. the range is all of $\mathbb{R}^3$: T is onto. (The Invertible Matrix Theorem: for a square matrix, one-to-one $\iff$ onto.)

Math 208 Final, Winter 2024 — Problem 5

(1) p_A has degree 4, so $\boxed{n = 4}$; its roots are the eigenvalues $\boxed{-1, 0, 1, 2}$.

(2) **No.** 0 is an eigenvalue, so A is not invertible; its rank is less than 4 and its rows cannot span $\mathbb{R}^4$.

(3) If $A\vec{v} = \lambda\vec{v}$ then $(A + 2I)\vec{v} = (\lambda + 2)\vec{v}$, so $A + 2I$ has eigenvalues 1, 2, 3, 4. None is 0, so $A + 2I$ is invertible: $\boxed{\text{rank}(A + 2I) = 4}$.

(4) $A - I$ has eigenvalues $-2, -1, 0, 1$. Its null space is the eigenspace E_1 of A ; since $\lambda = 1$ has algebraic multiplicity 1, $\dim E_1 = 1$. So $\text{nullity}(A - I) = 1$ and $\boxed{\text{rank}(A - I) = 4 - 1 = 3}$.

(5) **Yes.** A is 4×4 with 4 distinct eigenvalues; eigenvectors for distinct eigenvalues are linearly independent, so there are 4 independent eigenvectors.

Math 208 Final, Winter 2024 — Problem 6

$\text{rank}(A^3) < 3 \iff A^3$ is not invertible $\iff \det(A^3) = (\det A)^3 = 0 \iff \det A = 0$. Expand along the first row:

$$\det A = a \det \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix} - 3 \det \begin{bmatrix} 1 & 1 \\ 0 & a \end{bmatrix} + 0 = a(a^2 - 1) - 3a = a^3 - 4a = a(a - 2)(a + 2).$$

$$\boxed{a = 0, 2, -2}.$$

Check: at $a = 2$, $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $(3, -2, 1)$ is in its null space: $A(3, -2, 1) = (6 - 6, 3 - 4 + 1, -2 + 2) = \vec{0}$.

Math 208 Final, Winter 2024 — Problem 7

(1) Expand along the first row:

$$\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 3 & -\lambda \end{bmatrix} = (2 - \lambda) \det \begin{bmatrix} -\lambda & 0 \\ 3 & -\lambda \end{bmatrix} = (2 - \lambda)\lambda^2.$$

So $\boxed{\lambda_1 = 2, \lambda_2 = 0}$ ($\lambda_2 = 0$ with algebraic multiplicity 2).

(2) $\lambda_1 = 2$: row reduce $A - 2I$.

$$\begin{aligned} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 3 & -2 \end{bmatrix} &\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & -2 \end{bmatrix} \\ &\xrightarrow{-\frac{1}{2}R_1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & -2 \end{bmatrix} \\ &\xrightarrow{R_3 - 3R_1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\ &\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \\ &\xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Basis: $\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}}.$

(3) $\lambda_2 = 0$: row reduce A itself.

$$\begin{aligned} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} &\xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \\ &\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &\xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So $x_1 = x_2 = 0$ and x_3 is free. Basis: $\boxed{\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$.

(4) No. The eigenspaces have dimensions $1 + 1 = 2 < 3$ (for $\lambda = 0$ the geometric multiplicity is 1 but the algebraic multiplicity is 2), so there is no basis of eigenvectors.

Math 208 Final, Winter 2024 — Problem 8

Call the columns of A $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$. The columns of B are $2\mathbf{c}_1$, $\mathbf{c}_3$ and $\mathbf{c}_2 + \mathbf{c}_1$. Use the column versions of the row-operation rules:

$$\begin{aligned}\det B &= \det[2\mathbf{c}_1 \quad \mathbf{c}_3 \quad \mathbf{c}_2 + \mathbf{c}_1] = 2 \det[\mathbf{c}_1 \quad \mathbf{c}_3 \quad \mathbf{c}_2 + \mathbf{c}_1] && \text{(factor 2 from column 1)} \\ &= 2 \det[\mathbf{c}_1 \quad \mathbf{c}_3 \quad \mathbf{c}_2] && \text{(subtract column 1 from column 3)} \\ &= -2 \det[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3] = -2 \det A = -2 && \text{(swap columns 2 and 3)}\end{aligned}$$

Scaling all 3 rows of B by 2 multiplies the determinant by 2^3 :

$$\det(2B) = 2^3 \det B = 8(-2) = \boxed{-16}.$$

Math 208 Final, Winter 2024 — Problem 9

A vector $\mathbf{x}$ is in $\text{Nul } A$ exactly when $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + x_4\mathbf{u}_4 + x_5\mathbf{u}_5 = \vec{0}$. So $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_5$ are linearly dependent exactly when $\text{Nul } A$ contains a nonzero vector with $x_3 = x_4 = 0$.

The third spanning vector is redundant: $(2, 4, 4, 5, 6) = 2(1, 0, 2, 0, 3) + (0, 4, 0, 5, 0)$. So every null vector has the form

$$\mathbf{x} = s \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 0 \\ 4 \\ 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} s \\ 4t \\ 2s \\ 5t \\ 3s \end{bmatrix}.$$

Requiring $x_3 = 2s = 0$ and $x_4 = 5t = 0$ gives $s = t = 0$, so $\mathbf{x} = \vec{0}$. No nontrivial relation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_5\mathbf{u}_5 = \vec{0}$ exists, hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_5$ are linearly independent.