

Math 208 Midterm 2, Spring 2025 (Sara Billey) — Problem 1

(a) Row reduce A :

$$\begin{aligned}
 \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 8 & 2 \\ 0 & 5 \end{bmatrix} &\xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 8 & 2 \\ 0 & 5 \end{bmatrix} \\
 &\xrightarrow{R_3-8R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & -6 \\ 0 & 5 \end{bmatrix} \\
 &\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 \\ 0 & -6 \\ 0 & 0 \\ 0 & 5 \end{bmatrix} \\
 &\xrightarrow{-\frac{1}{6}R_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 5 \end{bmatrix} \\
 &\xrightarrow{R_1-R_2, R_4-5R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
 \end{aligned}$$

Two pivots (one in each column): $\boxed{\text{rank } A = 2}$, and $\boxed{\text{nullity } A = 2 - 2 = 0}$ (A has 2 columns).

(b) $\text{rank } A^T = \text{rank } A = 2$ (row rank equals column rank). A^T is 2×4 , so $\boxed{\text{rank } A^T = 2, \text{ nullity } A^T = 4 - 2 = 2}$.

(c) Compute B^2 , then factor: $AB^2 - AB = A(B^2 - B)$.

$$B^2 = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 16 \\ 0 & 1 \end{bmatrix}, \quad B^2 - B = \begin{bmatrix} 0 & 8 \\ 0 & 0 \end{bmatrix},$$

$$AB^2 - AB = \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 8 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 8 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 16 \\ 0 & 0 \\ 0 & 64 \\ 0 & 0 \end{bmatrix}.$$

(d) **Not invertible:** it is 4×2 , not square (and its first column is zero). Its first column is $\vec{0}$, so

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} : \quad (AB^2 - AB)\mathbf{v} = \text{first column} = \vec{0}.$$

Math 208 Midterm 2, Spring 2025 (Sara Billey) — Problem 2

Row reduce C :

$$\begin{aligned}
 \begin{bmatrix} 3 & -7 & 1 & 1 & 0 & 6 \\ 0 & 0 & -2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} &\xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & -\frac{7}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 2 \\ 0 & 0 & -2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \\
 &\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -\frac{7}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 & 0 & 0 \end{bmatrix} \\
 &\xrightarrow{R_1 - \frac{1}{3}R_2, R_3 + 2R_2} \begin{bmatrix} 1 & -\frac{7}{3} & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 & 2 & 2 \end{bmatrix} \\
 &\xrightarrow{\frac{1}{4}R_3} \begin{bmatrix} 1 & -\frac{7}{3} & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
 &\xrightarrow{R_1 - \frac{1}{3}R_3} \begin{bmatrix} 1 & -\frac{7}{3} & 0 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}
 \end{aligned}$$

Pivots are in columns 1, 3, 4.

(a) The pivot columns of the *original* C form a basis of $\text{col}(C)$:

$$\left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \right\} \quad (\text{so } \text{col}(C) = \mathbb{R}^3).$$

(b) Row operations don't change the row space, so the nonzero rows of the reduced matrix are a basis:

$$\left\{ (1, -\frac{7}{3}, 0, 0, -\frac{1}{2}, \frac{3}{2}), (0, 0, 1, 0, 1, 1), (0, 0, 0, 1, \frac{1}{2}, \frac{1}{2}) \right\}.$$

(c) Free variables x_2, x_5, x_6 . From the reduced rows: $x_1 = \frac{7}{3}x_2 + \frac{1}{2}x_5 - \frac{3}{2}x_6$, $x_3 = -x_5 - x_6$, $x_4 = -\frac{1}{2}x_5 - \frac{1}{2}x_6$. Setting one free variable at a time and clearing fractions:

$$\text{Nul}(C) = \text{span} \left\{ \begin{bmatrix} 7 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

Check: each vector gives $C\mathbf{v} = \vec{0}$; e.g. for $(1, 0, -2, -1, 2, 0)$: $(3 - 2 - 1 + 0, 4 - 4, -2 + 2) = \vec{0}$.

Math 208 Midterm 2, Spring 2025 (Sara Billey) — Problem 3

(a) Row operations do not change linear relations among the columns. B has pivots in columns 1 and 3, so $\{\mathbf{a}_1, \mathbf{a}_3\}$ is a basis of $\text{col}(A)$ and $\dim \text{col}(A) = 2$. Rule: a set of vectors in $\text{col}(A)$ is a basis exactly when it has 2 vectors and they are linearly independent.

(b) Read the relations off B : $\mathbf{b}_2 = \vec{0}$, so $\mathbf{a}_2 = \vec{0}$. Writing $\mathbf{b}_4 = c_1\mathbf{b}_1 + c_3\mathbf{b}_3$: the second entry gives $2c_3 = -1$, so $c_3 = -\frac{1}{2}$; the first gives $3c_1 - c_3 = 5$, so $c_1 = \frac{3}{2}$. Hence $\mathbf{a}_4 = \frac{3}{2}\mathbf{a}_1 - \frac{1}{2}\mathbf{a}_3$.

- $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$: **NO**, because it contains $\mathbf{a}_2 = \vec{0}$ (and has 3 vectors).
- $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$: **NO**, because 3 vectors in a 2-dimensional space are dependent ($\mathbf{a}_4 = \frac{3}{2}\mathbf{a}_1 - \frac{1}{2}\mathbf{a}_3$).
- $\{\mathbf{a}_1 + 2\mathbf{a}_4, \mathbf{a}_3 + \mathbf{a}_4\}$: these are $4\mathbf{a}_1 - \mathbf{a}_3$ and $\frac{3}{2}\mathbf{a}_1 + \frac{1}{2}\mathbf{a}_3$. Their coordinates in the basis $\{\mathbf{a}_1, \mathbf{a}_3\}$ give $\det \begin{bmatrix} 4 & \frac{3}{2} \\ -1 & \frac{1}{2} \end{bmatrix} = 2 + \frac{3}{2} = \frac{7}{2} \neq 0$: **YES**, two independent vectors.
- $\{\mathbf{a}_1, \mathbf{a}_3 + 2\mathbf{a}_4\}$: $\mathbf{a}_3 + 2\mathbf{a}_4 = \mathbf{a}_3 + 3\mathbf{a}_1 - \mathbf{a}_3 = 3\mathbf{a}_1$, a multiple of $\mathbf{a}_1$: **NO**.

Math 208 Midterm 2, Spring 2025 (Sara Billey) — Problem 4

(a) The columns of A are the images of e_1, e_2, e_3 :

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}, \quad Ae_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Ae_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad Ae_3 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \checkmark$$

(multiplying A by e_j picks out column j).

(b) Row reduce A :

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} &\xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -3 \end{bmatrix} \\ &\xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & \frac{3}{2} \end{bmatrix} \\ &\xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{3}{2} \end{bmatrix} \end{aligned}$$

Free variable x_3 : $x_1 = -\frac{3}{2}x_3$, $x_2 = -\frac{3}{2}x_3$. Clearing fractions, $\ker T$ has basis $\left\{ \begin{bmatrix} -3 \\ -3 \\ 2 \end{bmatrix} \right\}$. Two pivots, so

rank = 2 and the range is all of $\mathbb{R}^2$, with basis $\{(1, 1), (1, -1)\}$ (two independent columns).

Check: $A(-3, -3, 2) = (-3 - 3 + 6, -3 + 3 + 0) = (0, 0)$.

(c) The image of the cube is $\{x(1, 1) + y(1, -1) + z(3, 0) : 0 \leq x, y, z \leq 1\}$. The 8 corners map to $(0, 0), (1, 1), (1, -1), (3, 0), (2, 0), (4, 1), (4, -1), (5, 0)$. The image is the filled hexagon with vertices, in order, $(0, 0), (1, -1), (4, -1), (5, 0), (4, 1), (1, 1)$; the images $(3, 0)$ and $(2, 0)$ of two corners lie inside it. It is symmetric about the x -axis and spans $0 \leq x \leq 5, -1 \leq y \leq 1$.

Math 208 Midterm 2, Spring 2025 (Sara Billey) — Problem 5

Put the grid vectors in matrices (as columns): old $O = \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}$, new $W = \begin{bmatrix} -0.5 & 0.3 \\ 0.7 & 0.4 \end{bmatrix}$. A valve at old grid coordinates $\mathbf{c}_{\text{old}}$ sits at the point $O\mathbf{c}_{\text{old}}$, and its new coordinates satisfy $W\mathbf{c}_{\text{new}} = O\mathbf{c}_{\text{old}}$.

(a) $\det W = (-0.5)(0.4) - (0.3)(0.7) = -0.2 - 0.21 = -0.41$, so

$$W^{-1} = \frac{1}{-0.41} \begin{bmatrix} 0.4 & -0.3 \\ -0.7 & -0.5 \end{bmatrix},$$

$$\begin{aligned} W^{-1}O &= \frac{1}{-0.41} \begin{bmatrix} 0.4(-0.2) - 0.3(0.1) & 0.4(0.1) - 0.3(0.1) \\ -0.7(-0.2) - 0.5(0.1) & -0.7(0.1) - 0.5(0.1) \end{bmatrix} \\ &= \frac{1}{-0.41} \begin{bmatrix} -0.11 & 0.01 \\ 0.09 & -0.12 \end{bmatrix}. \end{aligned}$$

$$\boxed{\mathbf{c}_{\text{new}} = \frac{1}{41} \begin{bmatrix} 11 & -1 \\ -9 & 12 \end{bmatrix} \mathbf{c}_{\text{old}}}.$$

(b)

$$\frac{1}{41} \begin{bmatrix} 11(20) - 1(40) \\ -9(20) + 12(40) \end{bmatrix} = \frac{1}{41} \begin{bmatrix} 180 \\ 300 \end{bmatrix} = \boxed{\left(\frac{180}{41}, \frac{300}{41} \right) \approx (4.39, 7.32)}.$$

Check: both describe the same point: $O(20, 40) = (-4 + 4, 2 + 4) = (0, 6)$, and $W(\frac{180}{41}, \frac{300}{41}) = \frac{1}{41}(-90 + 90, 126 + 120) = (0, 6)$.

Bonus. (a) *Always:* $\vec{0}$ is in both; if $\mathbf{u}, \mathbf{w}$ are in both, so are $\mathbf{u} + \mathbf{w}$ and $c\mathbf{u}$ (each subspace is closed). (b) *Only when one contains the other* ($U \subseteq V$ or $V \subseteq U$). Otherwise pick $\mathbf{u} \in U$ not in V and $\mathbf{v} \in V$ not in U : if $\mathbf{u} + \mathbf{v}$ were in U then $\mathbf{v} = (\mathbf{u} + \mathbf{v}) - \mathbf{u} \in U$, a contradiction, and similarly it isn't in V , so the union is not closed under addition.