

Math 208 Midterm 2, Winter 2024 (Yang Zhang) — Problem 1

The echelon form B has pivots in columns 1 and 2.

(1) Row operations keep the row space, so the nonzero rows of B are a basis: $\{(1, -1, 1, 1), (0, 1, 2, 1)\}$, and $\dim \text{row}(A) = 2$.

(2) The pivot columns of A itself (columns 1 and 2) are a basis: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} \right\}$, and $\dim \text{col}(A) = 2$.

(3) $\text{rank}(A) = 2$ (the number of pivots).

(4) Solve $B\mathbf{x} = \vec{0}$ by back substitution, with $x_3 = s$, $x_4 = t$ free. Row 2: $x_2 = -2s - t$. Row 1: $x_1 = x_2 - x_3 - x_4 = (-2s - t) - s - t = -3s - 2t$.

$$\mathbf{x} = s \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{Nul}(A) = \text{span} \left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \text{nullity}(A) = 2.$$

Check: $A(-3, -2, 1, 0) = (-3 + 2 + 1, -6 + 4 + 2, -3 + 3) = \vec{0}$ and $A(-2, -1, 0, 1) = (-2 + 1 + 1, -4 + 2 + 2, -2 + 2) = \vec{0}$; also $\text{rank} + \text{nullity} = 2 + 2 = 4$ columns.

Math 208 Midterm 2, Winter 2024 (Yang Zhang) — Problem 2

(1) **No.** A basis of $\mathbb{R}^3$ has exactly 3 vectors; any 4 vectors in $\mathbb{R}^3$ are linearly dependent.

(2) Row reduce $[A \mid I]$:

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array} \right] &\xrightarrow{R_2-R_1, R_3-R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array} \right] \\ &\xrightarrow{R_1-R_2, R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right] \\ &\xrightarrow{R_2-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right] \end{aligned}$$

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Check: $AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

(3)

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}.$$

Check: $2\mathbf{u}_1 - \mathbf{u}_2 + 0\mathbf{u}_3 = (2-1, 2-2, 2-2) = (1, 0, 0) = \mathbf{b}.$

Math 208 Midterm 2, Winter 2024 (Yang Zhang) — Problem 3

(1) The plane $x - y + 2z = 0$ has y, z free with $x = y - 2z$, so a basis is $(1, 1, 0)$ ($y = 1, z = 0$) and $(-2, 0, 1)$ ($y = 0, z = 1$). Use them as columns, padding with zero columns to get a 3×4 matrix:

$$T(\mathbf{x}) = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}.$$

The range of T is the span of the columns, which is $\text{span}\{(1, 1, 0), (-2, 0, 1)\}$, exactly the plane. T is not one-to-one because $T(\mathbf{e}_3) = \vec{0}$ with $\mathbf{e}_3 \neq \vec{0}$.

Check: $(1, 1, 0)$: $1 - 1 + 0 = 0$; $(-2, 0, 1)$: $-2 - 0 + 2 = 0$.

(2) Multiply both sides of $(D + XB)^{-1}XA = C$ on the left by $D + XB$:

$$XA = (D + XB)C = DC + XBC \implies XA - XBC = DC \implies X(A - BC) = DC \implies \boxed{X = DC(A - BC)^{-1}}.$$

Math 208 Midterm 2, Winter 2024 (Yang Zhang) — Problem 4

(1) $T(\mathbf{x}) = \mathbf{x} \iff (A - I)\mathbf{x} = \vec{0}$. Row reduce $A - I$:

$$\begin{aligned} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} &\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ &\xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So $x_1 = 0$, $x_3 = 0$ and x_2 is free: $S = \text{span}\{(0, 1, 0)\}$, $\dim S = 1$.

Check: $A(0, 1, 0) = (0, 1, 0)$ (the second column of A).

(2) Row 3 has a single nonzero entry, $a_{34} = 1$, with sign $(-1)^{3+4} = -1$. Expand along row 3, then along the last row of the 3×3 minor:

$$\det A = -1 \cdot \det \begin{bmatrix} -1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = -\left(1 \cdot \det \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\right) = -(0 + 1) = -1.$$

And $\det(A^{-1}) = \frac{1}{\det A}$:

$$\boxed{\det A = -1, \quad \det(A^{-1}) = -1}.$$

Math 208 Midterm 2, Winter 2024 (Yang Zhang) — Problem 5

Let $\mathbf{v}_1, \dots, \mathbf{v}_4$ be the four inputs. Row reduce $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 3 \\ 1 & -1 & -1 & 3 \end{bmatrix} \xrightarrow{R_2-R_1, R_3-R_1, R_4-R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & -2 & 0 & 2 \\ 0 & -2 & -2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & 1 \\ 0 & -2 & -2 & 2 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & -2 & -2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1-R_2, R_4+2R_2} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1-R_3, R_4+2R_3} \begin{bmatrix} 1 & 0 & 0 & \frac{5}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{-R_4} \begin{bmatrix} 1 & 0 & 0 & \frac{5}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 - \frac{5}{2}R_4, R_2 + R_4, R_3 + \frac{1}{2}R_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Four pivots, so the inputs are linearly independent and form a basis of $\mathbb{R}^4$.

Their images are $(0, -1, 0, 0)$, $(0, 0, 0, 1)$, $(0, 0, -1, 0)$, $(1, 0, 0, 0)$, which are $\pm$ the four standard basis vectors, so they are linearly independent too.

Now suppose $T(\mathbf{x}) = \vec{0}$. Write $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4$ (possible since the $\mathbf{v}_i$ form a basis). Then $\vec{0} = c_1T(\mathbf{v}_1) + \cdots + c_4T(\mathbf{v}_4)$, and independence of the images forces $c_1 = c_2 = c_3 = c_4 = 0$, so $\mathbf{x} = \vec{0}$. The kernel is $\{\vec{0}\}$: **yes, T is one-to-one.**